RIGID JORDAN TUPLES

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Abstract. In a 1996 book, Katz introduced some remarkable objects in arithmetic geometry, rigid local systems. He gave an inductive classification of these systems, and asked for more explicit results concerning this classification.

Here we use much more elementary language, speaking of rigid Jordan tuples rather than rigid local systems. We present a substantially streamlined version of Katz's classification. We provide some explicit results of the type Katz asked for.

Contents

1. Overview

Fix for this entire paper an algebraically closed field E. Consider unordered z-tuples $j = \{j_1, \ldots, j_z\}$ of non-central conjugacy classes in $GL_n(E)$, $z \geq 0$ and $n \geq 1$ being integers. Impose the determinant condition

$$\det(j_1) \cdots \det(j_z) = 1. \tag{1.1}$$

Also impose a rigidity condition on the dimension of these conjugacy classes,

$$\dim(j_1) + \cdots + \dim(j_z) = 2(n^2 - 1). \tag{1.2}$$

We call such $\{j_1, \ldots, j_z\}$ *rigid Jordan z-tuples of rank n*. Here we use the word "Jordan" because we will be describing each j_i in terms of Jordan canonical forms.

Let $(j_1, \ldots, j_z)$ be an ordered list of non-central conjugacy classes in $GL_n(E)$ satisfying Conditions (1.1) and (1.2). Let $V(j_1, \ldots, j_z)$ be the subset of $j_1 \times \cdots \times j_z$ consisting of matrix tuples $(g_1, \ldots, g_z)$ with $g_1 \cdots g_z = 1$ and $\langle g_1, \ldots, g_z \rangle$ acting irreducibly on E^n. The group $PGL_n(E)$ acts on $V(j_1, \ldots, j_n)$ by simultaneous conjugation. A naive dimension count suggests that there are only finitely many orbits. It is elementary that the number of orbits is independent of the ordering

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of the j_i. A cohomological argument says that in fact the number of orbits is either zero or one [10]. In the latter case, we say that $\{j_1, \ldots, j_z\}$ is *realizable.* The problem addressed in this paper is the explicit description of realizable rigid Jordan tuples.

A rigid local system in the sense of Katz [7] gives rise to a realizable rigid Jordan tuple in our sense. Information has been lost in the passage from rigid local systems to realizable rigid Jordan tuples. For example, a finite subset S of the Riemann sphere $\mathbf{C} \cup \{\infty\}$ has been replaced by the number $z = |S|$. However the information lost is trivial from the point of view of classification, meaning that our classification of realizable rigid Jordan tuples immediately translates back into a classification of rigid local systems.

In §2 we introduce a formalism for conveniently dealing with Jordan canonical forms. In §3 we present Katz's remarkable inductive algorithm for classifying rigid local systems, recast into the setting rigid Jordan tuples. Katz uses complicated algebro-geometric language necessary for his proof that his classification is correct. We use radically simpler language, insufficient for a presentation of Katz's proof, but ideal for pursuing explicit classification results. In particular, our formalism exploits the fact that rigid Jordan tuples come in families, and it makes sense to talk about realizability at the level of families. At the level of classifying realizable families, only the very simplest parts of the algorithm are involved. The coefficient field E does not play a role at all, and the formalism centers on partitions. The realizable rigid Jordan tuples within a given realizable family naturally form an irreducible affine variety over E. In our terminology, the dimension of a family is one less than its length ℓ.

With respect to rigid local systems, Katz wrote [7, page 9] of "a fascinating bestiary waiting to be compiled." In §4-9 we present such a compilation at the level of rigid Jordan tuples. In dimension ≤ 11, the results can be summarized as follows.

Proposition 1.1. *The number $|rRPT_{n,\ell}|$ of realizable families of rank n length ℓ rigid Jordan tuples for $2 \leq n \leq 11$ is as in Table 1.1.*

TABLE 1.1.

$n \setminus \ell$	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	1																		
3			2																
4			1	3	2														
5			1	2	6		2												
6			1	4	9	12			2										
7			1	3	12	14	12				2								
8			1	5	19	32	25	12					2						
9			1	6	24	47	53	12	12						2				
10			1	7	33	84	96	65	6	12							2		
11			1	7	42	106	143	96	32		12								2

In §4, we indicate how Proposition 1.1 can be proved by a computer search. In §5 we list out and label the families in ranks $n \leq 6$. For example, the "realizable rigid partition tuple"

$$\{22, 211, 1111\}, \tag{1.3}$$

which we denote A_4, indexes one of the three families with $n = 4$ and $\ell = 9$. Here the length ℓ is the number of parts of the joint partition 222111111.

In §6 we examine the inner structure of the families with $n \leq 6$ and $z = 3$. For example, members of A_4, in our notation, have the form

$$\{a_1^2 a_2^2, B^2 b_1 b_2, c_1 c_2 c_3 c_4\}. \tag{1.4}$$

Here a_1, a_2, B, b_1, b_2, c_1, c_2, c_3, c_4 are in $E^\times$ and satisfy the determinant condition

$$a_1^2 a_2^2 \cdot B^2 b_1 b_2 \cdot c_1 c_2 c_3 c_4 = 1 \tag{1.5}$$

and the inequalities

$$\begin{array}{rlll} a_i B c_j \neq 1, & i = 1, 2, & j = 1, 2, 3, 4 & \\ a_1 a_2 B b_i c_j c_k \neq 1, & i = 1, 2, & j, k = 1, 2, 3, 4 & j \neq k. \end{array} \tag{1.6}$$

In (1.4)-(1.6), a_1, ... , c_4 represent eigenvalues, and the Jordan block structure is obtained by dualizing the exponents, as explained in §2. The internal structure of all other families looks qualitatively similar: a single determinant condition like (1.5), explaining why the dimension of the family is $\ell - 1$, and supplemental multiplicative inequalities like (1.6).

Note from Table 1.1 that, for $6 \leq n \leq 11$ at least, the maximal possible ℓ is $2n + 2$, achieved by two families. The next largest possible ℓ is $n + 5$, coming from twelve families. The smallest possible ℓ is 8, achieved once. In §7, 8, 9 we discuss these *maximal*, *submaximal*, and *extreme minimal* cases respectively, proving that the patterns continue indefinitely.

The 2's in the maximal case come from what we call the hypergeometric family H_n and the Pochhammer family P_n. These two families have $z = 3$ and $z = n + 1$ respectively. They can reasonably be considered classical, and §7 may serve for some readers as an illustration of how our formalism looks in a familiar context.

In our discussion in §8 of the submaximal case, the hypergeometric and Pochhammer families reappear because of the inductive nature of the classification. In this section, we also prove that the gap between the 12's and the 2's in Table 1.1 continues indefinitely.

Finally, in §9, we identify some families with $\ell = 9, 10, 11$ which we also call *minimal*. We discuss these together with the extreme minimal case.

The principal interest in our subject is that it forms the combinatorial core of a very rich motivic theory [7, Chapter 8]. Rigid matrix tuples $(g_1, \ldots, g_z) \in V(j_1, \ldots, j_z)$ arise as underlying monodromy representations. For the family H_n, hypergeometric functions ${}_{n-1}F_n$ arise as period integrals. Similarly for the family P_n, Pochhammer functions arise as period integrals. The remaining realizable families give rise to similar functions which have just begun to be studied. There are many arithmetic questions which have not been answered even for the classical families H_n and P_n.

Other authors have built on the work of Katz in different directions, sometimes referring to the general subject as the Deligne-Simpson problem. We refer the reader to recent papers of Belkale, Crawley-Boevey, Dettweiler, Gleizer, Kostov, Reiter, Strambach, and Völklein. Also there is a connection with the circle of ideas involved in the recent solution of Horn's conjecture, see e.g. [8].

2. Jordan formalism

For n a positive integer, let P_n be the set of partitions of n. We write out partitions by listing their parts: $\lambda = \{\lambda_1, \ldots, \lambda_\ell\}$. Order does not matter, and we usually choose to list parts in decreasing order. The rank of λ is simply the number $n = |\lambda| = \sum \lambda_k$.

The *maxmult* of $\lambda \in P_n$ is simply the largest part $m(\lambda)$. The *length* of λ is the number $\ell(\lambda)$ of parts. The *norm* of λ is the quantity $||\lambda|| = \sum \lambda_k^2$. Note that $|\lambda| \equiv ||\lambda||$ modulo two. We often omit braces and commas when the meaning is clear. Thus $P_4 = \{\{1,1,1,1\}, \{2,1,1\}, \{2,2\}, \{3,1\}, \{4\}\} = \{1111, 211, 22, 31, 4\}$.

In (2.2) below, we make use of the standard duality $t : P_n \to P_n$ which interchanges maxmult m and length ℓ. Graphically, one can think of t as transpose, e.g.

$$(3,1)^t = \begin{pmatrix} \bullet, & \bullet \\ \bullet & \\ \bullet & \end{pmatrix}^t = \begin{pmatrix} \bullet, & \bullet, & \bullet \\ \bullet & & \end{pmatrix} = (2,1,1).$$

Algebraically, the dual λ^t of a given partition λ is defined by letting λ_k^t be the number of parts of λ of size $\geq k$.

For $\lambda \in P_n$, define J_λ to be the set of formal symbols $\{a_1^{\lambda_1}, \ldots, a_\ell^{\lambda_\ell}\}$ with $a_k \in E^\times$. So J_λ is a copy of the ℓ-dimensional torus $(E^\times)^\ell$ modded out by a product of symmetric groups. E.g. J_{321} is simply a three-dimensional torus, the general element being $\{a^3, b^2, c\}$ for a unique $(a,b,c) \in E^{\times 3}$. On the other hand J_{222} is the torus modulo S_3, as the general element can be expressed as $\{a^2, b^2, c^2\}$ for six different choices of the ordered triple (a,b,c). Here the exponents are part of the formalism, and not an indication of multiplication.

Put

$$J_n := \coprod_{\lambda \in P_n} J_\lambda. \tag{2.1}$$

If $j \in J_\lambda$ we say that λ is the *centralizer partition* of j.

Let $g \in GL_n(E)$ be a matrix. For $a \in E^\times$, let n_a be the dimension of the generalized eigenspace of a, i.e. the dimension of the kernel of $(g - aI_n)^n$. Let μ_a be the partition of n_a giving the sizes of the Jordan blocks belonging to a. Put

$$\lambda_a = \mu_a^t. \tag{2.2}$$

A good case to keep in mind is the case when g acts just by the scalar a on the generalized eigenspace of a. Then the corresponding partitions of n_a are $\mu_a = 11\cdots11$ and $\lambda_a = n_a$. The μ_a will not play any further role.

The theory of Jordan canonical forms identifies the set of conjugacy classes in the group $GL_n(E)$ with the set J_n, via

$$[g] = \{a_1^{\lambda_1}, \ldots, a_\ell^{\lambda_\ell}\}.$$

Here, for a given $a \in E^\times$, the exponents on a all together form λ_a. The largest part of λ_a is denoted $d([g], a)$, and called the *drop* of $[g]$ with respect to a.

In the Jordan formalism just set up, the natural action of $E^\times$ on conjugacy classes is given by the formula

$$a\{a_1^{\lambda_1},\ldots,a_\ell^{\lambda_\ell}\} = \{(aa_1)^{\lambda_1},\ldots,(aa_\ell)^{\lambda_\ell}\}. \tag{2.3}$$

Similarly,

$$\det\left(\{a_1^{\lambda_1},\ldots,a_\ell^{\lambda_\ell}\}\right) = a_1^{\lambda_1}\cdots a_\ell^{\lambda_\ell} \tag{2.4}$$

expresses the determinant of a conjugacy class; on the right of (2.4), exponents are indicating multiplication. Since the determinant condition (1.1) plays a central role in this paper, so does (2.4).

The centralizer in $GL_n(E)$ of a matrix g with class $(a_1^{\lambda_1},\ldots,a_\ell^{\lambda_\ell})$ has dimension

$$\mathrm{centdim}(\{a_1^{\lambda_1},\ldots,a_\ell^{\lambda_\ell}\}) = \sum_{k=1}^{\ell}\lambda_k^2 = ||\lambda||. \tag{2.5}$$

In fact, if $g \in J_\lambda$ is semisimple then the centralizer has the form $\prod_k GL_{\lambda_k}(E)$. The dimension of the conjugacy class is

$$\dim(\{a_1^{\lambda_1},\ldots,a_\ell^{\lambda_\ell}\}) = n^2 - ||\lambda||. \tag{2.6}$$

Since the rigidity condition (1.2) plays a central role in this paper, so do (2.5) and (2.6).

3. Katz's algorithm

The notation set up in the previous section will generally be used henceforth with an appended index. Thus $\lambda_i = \{\lambda_{i,1},\ldots,\lambda_{i,\ell_i}\}$ now typically denotes a single partition of n. Similarly j_i typically denotes a single class in $GL_n(E)$. By replacing each j_i by the partition λ_i indexing its component (2.1), one associates to a *rigid Jordan tuple* $\{j_1,\ldots,j_z\}$ a *rigid partition tuple* $\{\lambda_1,\ldots,\lambda_z\}$.

Throughout this paper we systematically use the following abbreviations

r	realizable	R	rigid	J	Jordan	T	tuple
p	plausible			P	partition.		

Thus RJ_zT_n denotes the set of rigid Jordan z-tuples of rank n. Similarly $RJT = \coprod RJ_zT_n$ is the set of all rigid Jordan tuples. We will be focused on a diagram

$$\begin{array}{ccccc} rRJT & \subset & pRJT & \subset & RJT \\ \downarrow & & \downarrow & & \downarrow \\ rRPT & \subset & pRPT & \subset & RPT \end{array} \tag{3.1}$$

In this diagram, each vertical arrow can be viewed as the passage to connected components. The word "family" will be used as a synonym for "rigid partition tuple." Thus our subject is the explicit description of $rRJT$ and we focus mostly on the explicit description of the set $rRPT$ of realizable families.

For $\lambda = \{\lambda_1,\ldots,\lambda_z\}$ with λ_i in P_n define its *joint partition* to be

$$\mu = \lambda_1 \coprod \cdots \coprod \lambda_z \in P_{zn}. \tag{3.2}$$

Whether or not λ is in RPT depends only on μ, as the rigidity condition (1.2) becomes, via (2.5) and (2.6),

$$||\mu|| = (z-2)n^2 + 2. \tag{3.3}$$

Write $\mu = \{\mu_1, \mu_2, \mu_3, \cdots\}$ with $\mu_i \geq \mu_{i+1}$. We say that $\lambda \in RP_zT_n$ is *plausible* iff $n = 1$ or

$$\mu_1 + \mu_2 + \cdots + \mu_{n-1} \leq (z-2)n < \mu_1 + \mu_2 + \cdots + \mu_{n-1} + \mu_n. \tag{3.4}$$

Let $j \in RJ_zT_n$. We say that j is plausible if its centralizer partition tuple $\lambda \in RP_zT_n$ is plausible and moreover

$$\text{If } a_1 \cdots a_z = 1, \text{ then } d(j_1, a_1) + \cdots + d(j_z, a_z) \leq (z-2)n. \tag{3.5}$$

We denote the set of plausible rigid Jordan z-tuples of rank n by pRJ_zT_n.

It is a relatively elementary fact that $rRJT \subseteq pRJT$. Similarly $rRPT \subseteq pRPT$. The remaining theoretical issue is to distinguish realizability from mere plausibilty. Here one proceeds inductively on n. The base of the induction is the case $n = 1$. This $n = 1$ case is quite degenerate: RJT_1 has one element $\{\{\}\}$ which is plausible and moreover realizable; similarly RPT_1 has one element $\{\{\}\}$ which is plausible and realizable.

On the partition level the induction is very simple. A marking v on $\lambda \in pRP_zT_n$ is a part v_i in each λ_i such that the sum $d(v) = v_1 + \cdots + v_z - (z-2)n$ is positive. The derivative of λ with respect to v is denoted $\partial_v\lambda$. Here $(\partial_v\lambda)_i$ is obtained from λ_i by replacing one v_i in λ_i by $v_i - d(v)$ and dropping scalar partitions. The Katz classification is the following.

(3.6) *Let $\lambda \in pRPT$. Let v be a marking on λ. Then $\partial_v\lambda$ realizable implies λ realizable.*

Since derivation reduces rank by at least one, this statement is indeed an effective classification: after enough derivations one has reached either a non-plausible tuple or a tuple already known to be realizable.

At this level of partition tuples, one has a canonical marking $v_{\max}$. Here $v_{\max,i}$ is just the largest part of λ_i. Here is a sample computation:

$$\lambda = \begin{matrix} 721 \\ 631 \\ 7111 \\ 55 \end{matrix} \xrightarrow{5} \begin{matrix} 221 \\ 311 \\ 2111 \end{matrix} \xrightarrow{2} \begin{matrix} 21 \\ 111 \\ 111 \end{matrix} \xrightarrow{1} \begin{matrix} 11 \\ 11 \\ 11 \end{matrix} \xrightarrow{1} \quad . \tag{3.7}$$

Each arrow indicates maximal derivation, the superscript indicating the associated drop in rank. All the displayed partition triples are plausible, and the last one is realizable; so the computation shows that λ is realizable.

On the Jordan-level the induction is more complicated, as one has to keep track of changing eigenvalues. Let $j \in J_\lambda$ with $\lambda \in pRPT$. A marking V on j is an element $V_i = a_i^{v_i}$ of each j_i such that 1) each v_i maximal for a_i; 2) v is a marking on λ. Automatically

$$\pi(V) := a_1 \cdots a_z$$

is not one by (3.5). Write $j_i = \{a_i^{v_i}, \mathrm{rest}_i\}$. Define the *derivative* of j with respect to the marking V to be $\partial_V j = \{(\partial_V j)_1, \ldots, (\partial_V j)_z\}$ with

$$(\partial_V j)_i = a_i\{(\pi(V)/a_i)^{v_i - d(v)}, \mathrm{rest}_i\} \tag{3.8}$$

The rigidity (1.2) and determinant (1.1) conditions are satisfied, so that $\partial_V j \in RJT_{n-d(v)}$; it may or may not be plausible. The Katz classification here says

(3.9) *Let $j \in pRJT$. Let V be a marking for j. Then $\partial_V j$ realizable implies j realizable.*

Illustrations of (3.9) are given in §6, 7.

Katz's statement of his algorithm is more complicated than the statement we have given here. Here are four salient points, which should guide the diligent reader in checking that our restatement is correct. First, Katz carries out as much as possible without imposing the rigidity condition (1.2). Second, Katz's notation is purposely highly redundant as he emphasizes. For our purposes, this redundancy is not useful and so of his (r, m, e, E) we use only (r, e) (his r being our n). Third, Katz works with finite subsets D of some algebraically closed field K. A direct translation to language close to ours would be to identify $D \cup \{\infty\}$ with $\{1, \ldots, z\}$ and work with ordered z-tuples and allow scalar classes. Our passage to unordered tuples, disallowed scalar classes, and thus perhaps decreasing z is highly unnatural geometrically; however it is natural from the limited point of view of our classification questions. Fourth our operation of derivation involves middle-tensoring, middle-convoluting, and middle-tensoring again, this being essentially one iteration of Step II-Step VI of his algorithm [7, 6.4.1]. One advantage of combining these operations is that the elements of Katz's set D and the extra point ∞ are treated on the same footing, the distinction thus disappearing in our formalism.

4. $n \leq 11$: QUICK LOOK

The reader can quickly check that using (1.2), (2.5), and (2.6), that

$$
\begin{array}{rclcrcl}
RPT_1 & = & \{H_1\} & & H_1 & = & \{\} \\
 & & & & & & \\
RPT_2 & = & \{H_2\} & \text{where} & H_2 & = & \{11, 11, 11\} \\
 & & & & & & \\
RPT_3 & = & \{H_3, P_3\} & & H_3 & = & \{21, 111, 111\} \\
 & & & & P_3 & = & \{21, 21, 21, 21\}.
\end{array}
$$

Also in these cases the plausibility condition (3.4) and the realization condition (3.9) are satisfied so that $rRPT_n = pRPT_n = RPT_n$. Note that H_1, with $z = 0$ is anomolous; all other members of RPT have $z \geq 3$. Note that it would also be natural to give a given element of $rRPT$ several names, e.g. $H_1 = P_1$ and $H_2 = P_2$.

The case $n = 4$ is quite manageable by hand too, but at some point soon thereafter it is only reasonable to use a computer. In fact, we have implemented two reasonable approaches. One, following our main text, lists out $pRPT$ and then selects $rRPT$ from these. The other, more in line with the proof of Theorem 8.1, starts from H_1 and at every step takes all legal antiderivatives; this method never sees non-realizable tuples. Either way, the code required is modest, about 40 lines. Table 1.1 summarizes the results. Run times are modest too, and so one could easily extend this table beyond $n = 11$.

There are other apparent general patterns in the computer data beyond the three that we pursue in Sections 7, 8, and 9. For example, one can expect that the blank space in slot $(n, \ell) = (11, 15)$ forms the tip of an infinite blank region under the submaximal diagonal of 12's. Similarly, when one sorts realizable rigid Jordan tuples by (n, ℓ, m), m being the maximal part of the joint partition, other presumably infinite blank regions appear.

5. $n \leq 6$: CLOSER LOOK

Of course the programs giving Table 1.1 give $rRPT_{n,\ell}$, not just $|rRPT_{n,\ell}|$. Tables 5.1 and 5.2 give $rRPT_{n,\ell}$ out through $n = 6$.

TABLE 5.1. The elements of $rRPT_{n,\ell}$, $n \leq 5$, and their maximal derivatives.

ℓ	λ	λ_1	λ_2	λ_3	$\cdots$			$\partial_{\max}\lambda$
	H_1							
6	H_2	11	11	11				H_1
8	H_3	21	111	111				H_2
8	P_3	21	21	21	21			H_1
10	H_4	31	1111	1111				H_3
10	P_4	31	31	31	31	31		H_1
9	A_4	22	211	1111				H_3
9	B_4	211	211	211				H_2
9	I_4	31	31	22	211			H_2
8	$\delta_{4,2}$	31	22	22	22			P_3
12	H_5	41	11111	11111				H_4
12	P_5	41	41	41	41	41	41	H_1
10	A_5	32	221	11111				A_4
10	B_5	311	221	2111				H_3
10	C_5	32	2111	2111				H_3
10	I_5	41	32	311	311			H_2
10	J_5	41	41	221	221			P_3
10	M_5	41	41	41	32	32		H_2
9	α_5	221	221	221				B_4
9	q_5	41	32	32	221			P_3
8	δ_5	32	32	32	32			P_3

In [7, page 165], Katz gave examples due to Deligne of plausible but non-realizable rigid Jordan tuples. These examples all have rank seven. It is natural to ask whether there are examples of lower rank. The next two sections show that there are no examples in ranks $n \leq 3$, but then examples in rank $n = 4$ belonging to the series A_4 and B_4.

One can also ask for the first examples of plausible but non-realizable rigid partition tuples. In fact for $n \leq 5$ there are none. For $n = 6$ there is one: $\{51, 33, 33, 3111\}$ is plausible but its maximal derivative $\{31, 31, 31, 1111\}$ is not.

Finally one can ask for the first examples of plausible but non-realizable rigid partition tuples, in the context $z = 3$. There are none for $n \leq 6$. For $n = 7$ there is one: $\{331, 331, 31111\}$ is plausible but its maximal derivative $\{311, 311, 11111\}$ is not. Deligne's plausible but non-realizable rigid Jordan tuples belong to this family.

Remarks. Tables 5.1 and 5.2 are useful in that they aid in working explicitly with examples. For example, consider the 120 element group $\tilde{A}_5$. It has a presentation

$$\tilde{A}_5 = \langle g_1, g_2, g_3, z | g_1 g_2 g_3 = 1;\ g_1^2 = g_2^3 = g_3^5 = z;\ z^2 = 1,\ z \text{ central}\rangle.$$

The group $\tilde{A}_5$ has nine irreducible complex representations ρ. Put $j_i = [\rho(g_i)]$. All the resulting Jordan triples $\{j_1, j_2, j_3\}$ are rigid, as one can tell from direct inspection or from (9.2) with ℓ_1, ℓ_2, ℓ_3 at most $2, 3, 5$ respectively. One can use the ATLAS to figure out the j_i explicitly, and hence the λ_i. The rigid partition triples arising are, in ATLAS order, H_1, H_3, H_3, A_4, A_5; H_2, H_2, A_4, γ_6.

Table 5.2. The elements of $rRPT_{6,\ell}$ and their maximal derivatives.

ℓ	λ	λ_1	λ_2	λ_3	$\cdots$				$\partial_{\max}\lambda$
14	H_6	51	111111	111111					H_5
14	P_6	51	51	51	51	51	51	51	51	H_1
11	A_6	33	321	111111						A_5
11	B_6	321	3111	3111						H_3
11	C_6	33	3111	21111						H_4
11	D_6	42	222	111111						A_5
11	E_6	42	2211	21111						A_4
11	F_6	411	222	21111						A_4
11	G_6	411	2211	2211						B_4
11	I_6	51	33	411	3111					H_3
11	J_6	51	51	222	2211					I_4
11	K_6	51	51	51	33	321				P_3
11	M_6	51	51	42	42	411				H_2
11	N_6	42	411	411	411					H_2
10	$\gamma_{6,6}$	33	222	21111						A_5
10	β_4	33	2211	2211						C_5
10	q_6	321	321	2211						B_4
10	r_6	321	222	3111						A_4
10	s_6	51	42	33	2211					I_4
10	t_6	51	42	321	321					P_3
10	u_6	51	33	411	222					I_4
10	v_6	42	33	411	411					H_3
10	w_6	51	51	42	42	33				P_3
9	$\alpha_{6,3}$	321	222	222						α_5
9	x_6	51	33	33	222					q_5
9	y_6	42	42	42	321					P_3
9	z_6	42	33	33	411					I_4
8	$\delta_{6,2}$	42	33	33	33					δ_5

In particular, Tables 5.1 and 5.2 single out certain covers of the projective line as having a motivic interpretation and, as a consequence, good reduction even at primes dividing the order of the monodromy group. See [9] for examples in the context $z = 3$.

6. $n \leq 6$, $z = 3$: internal structure

In this section, we completely describe the internal structure of all realizable families with $n \leq 6$ and $z = 3$.

Proposition 6.1. *Let λ be a realizable rigid partition triple of rank $n = 4$, 5, or 6. Let $j = \{j_1, j_2, j_3\} \in J_\lambda$. Then whether or not j is realizable is as described Table 6.1 Here subscripts i, j, k, l on the same symbol are required to be different, but are otherwise arbitrary.*

Proof. The proof consists in applying the Katz algorithm (3.9) in each case. Here we describe the computation in the last case $\lambda = \alpha_6$. Starting from α_6, we take

TABLE 6.1. Realizablility of elements of $RJT_{n,\ell}$ for $n = 4, 5, 6$

λ	j_1	j_2	j_3	*j is realizable iff it is plausible and the quantities below are different from* 1.	
H_4	A^3a	$b_1b_2b_3b_4$	$c_1c_2c_3c_4$		
A_4	$a_1^2a_2^2$	$B^2b_1b_2$	$c_1c_2c_3c_4$	$a_1a_2Bb_ic_jc_k$	
B_4	$A^2a_1a_2$	$B^2b_1b_2$	$C^2c_1c_2$	$Aa_iBb_jCc_k$	
H_5	A^4a	$b_1b_2b_3b_4b_5$	$c_1c_2c_3c_4c_5$		
A_5	A^3a^2	$B_1^2B_2^2b$	$c_1c_2c_3c_4c_5$	$AaB_1B_2c_ic_j$	
B_5	$A_1^2A_2^2a$	Bb_1b_2	$C^2c_1c_2c_3$	$A_1A_2Bb_iCc_j$	
C_5	A^3a^2	$B^2b_1b_2b_3$	$C^2c_1c_2c_3$	$AaBb_iCc_j$	
α_5	$A_1^2A_2^2a$	$B_1^2B_2^2b$	$C_1^2C_2^2c$	$A_1A_2B_1B_2C_ic$	$A_1A_2B_ibC_1C_2$
				$A_iaB_1B_2C_1C_2$	
H_6	A^5a	$b_1b_2b_3b_4b_5b_6$	$c_1c_2c_3c_4c_5c_6$		
A_6	$a_1^3b_1^3$	$\hat{B}^3B^2b$	$c_1c_2c_3c_4c_5c_6$	$a_1a_2\hat{B}Bc_ic_j$	
B_6	$\hat{A}^3A^2a$	$B^3b_1b_2b_3$	$C^3c_1c_2c_3$	$\hat{A}ABb_iCc_j$	
C_6	$a_1^3a_2^3$	$Bb_1b_2b_3$	$C^2c_1c_2c_3c_4$	$a_1a_2Bb_iCc_j$	
D_6	A^4a^2	$b_1^2b_2^2b_3^2$	$c_1c_2c_3c_4c_5c_6$	$A^2ab_1b_2b_3c_ic_jc_k$	
E_6	A^4a^2	$B_1^2B_2^2b_1b_2$	$C^2c_1c_2c_3c_4$	$AaB_1B_2Cc_i$	$A^2aB_1B_2b_ic_jc_k$
F_6	$a_1^2a_2^2a_3^2$	$B^4b_1b_2$	$C^2c_1c_2c_3c_4$	$a_1a_2a_3B^2b_iCc_jc_k$	
G_6	$A^4a_1a_2$	$B_1^2B_2^2b_1b_2$	$C_1^2C_2^2c_1c_2$	$Aa_iB_1B_2C_1C_2$	$A^2a_iB_1B_2b_jC_1C_2c_k$
γ_6	$a_1^3a_2^3$	$b_1^2b_2^2b_3^2$	$C^2c_1c_2c_3c_4$	$a_1a_2b_ib_jCc_k$	$a_1^2a_2b_1b_2b_3c_ic_j$
β_6	$a_1^3a_2^3$	$B_1^2B_2^2b_1b_2$	$C_1^2C_2^2c_1c_2$	$a_1a_2B_1B_2C_ic_j$	$a_1a_2B_ib_jC_1C_2$
				$a_1^2a_2B_1B_2b_iC_1C_2c_j$	
q_6	$a_1^2a_2^2a_3^2$	$\hat{B}^3B^2b$	$C^3c_1c_2c_3$	$a_ia_j\hat{B}BCc_k$	$a_1a_2a_3\hat{B}BbC^2c_i$
r_6	$\hat{A}^3A^2a$	$\hat{B}^3B^2b$	$C^3c_1c_2c_3$	$\hat{A}A\hat{B}BCc_i$	$\hat{A}A\hat{B}bCc_i$
				$\hat{A}a\hat{B}BCc_i$	
α_6	$a_1^2a_2^2a_3^2$	$b_1^2b_2^2b_3^2$	$\hat{C}^3c^2c$	$a_ia_jb_kb_l\hat{C}C$	$a_1a_2a_3b_1b_2b_3\hat{C}^2C$
				$a_1a_2a_3b_1b_2b_3\hat{C}C^2$	

three derivatives (3.8) successively, as indicated by the $\xrightarrow{d}$; here d indicates the drop in rank. In between, as indicated by $=$, we simplify by scalar multiplication (2.3). The $\bullet$s at each derivation step indicate the choice of marking; the $\circ$s on the next line indicate the slots corresponding to the previous $\bullet$s, to increase readability. In this computation, exponents are reserved to be part of the Jordan formalism, which explains why factors are often simply repeated.

$$\left\{\begin{array}{lrr} \{\bullet a_1^2, & a_2^2, & a_3^2\} \\ \{\bullet b_1^2, & b_2^2, & b_3^2\} \\ \{\bullet \hat{C}^3, & C^2, & c\} \end{array}\right\} = j$$

$$\xrightarrow{1} \quad \begin{array}{c} a_1 \\ b_1 \\ \hat{C} \end{array} \left\{\begin{array}{lrr} \{\circ(1/b_1\hat{C}), & a_2^2, & a_3^2\} \\ \{\circ(1/a_1\hat{C}), & b_2^2, & b_3^2\} \\ \{\circ(1/a_1b_1)^2, & C^2, & c\} \end{array}\right\}$$

$$= \quad \left\{\begin{array}{lrr} \{(a_1/b_1\hat{C}), & \bullet(a_1a_2)^2, & (a_1a_3)^2\} \\ \{(b_1/a_1\hat{C}), & \bullet(b_1b_2)^2, & (b_1b_3)^2\} \\ \{(\hat{C}/a_1b_1)^2, & \bullet(\hat{C}C)^2, & (\hat{C}c)\} \end{array}\right\} = j'$$

$$(6.1)\qquad \xrightarrow{1} \begin{array}{c}(a_1a_2)\\(b_1b_2)\\(\hat{C}C)\end{array} \left\{\begin{array}{lcr} \{a_1/b_1\hat{C}, & \circ 1/b_1b_2\hat{C}C, & (a_1a_3)^2\} \\ \{b_1/a_1\hat{C}, & \circ 1/a_1a_2\hat{C}C, & (b_1b_3)^2\} \\ \{(\hat{C}/a_1b_1)^2, & \circ 1/a_1a_2b_1b_2, & \hat{C}c\} \end{array}\right\}$$

$$= \left\{\begin{array}{lcr} \{a_1a_1a_2/b_1\hat{C}, & a_1a_2/b_1b_2\hat{C}C, & \bullet(a_1a_1a_2a_3)^2\} \\ \{b_1b_1b_2/a_1\hat{C}, & b_1b_2/a_1a_2\hat{C}C, & \bullet(b_1b_1b_2b_3)^2\} \\ \{\bullet(\hat{C}\hat{C}C/a_1b_1)^2, & \hat{C}C/a_1a_2b_1b_2, & \hat{C}\hat{C}Cc\} \end{array}\right\} = j''$$

$$\xrightarrow{2} \begin{array}{l}a_1a_1a_2a_3\\b_1b_1b_2b_3\\\hat{C}\hat{C}C/a_1b_1\end{array} \left\{\begin{array}{llr} \{a_1a_1a_2/b_1\hat{C}, & a_1a_2/b_1b_2\hat{C}C & \circ\} \\ \{b_1b_1b_2/a_1\hat{C}, & b_1b_2/a_1a_2\hat{C}C & \circ\} \\ \{\circ & \hat{C}C/a_1a_2b_1b_2, & \hat{C}\hat{C}Cc\} \end{array}\right\}$$

$$= \left\{\begin{array}{lr} \{a_1a_1a_1a_1a_2a_2a_3/b_1\hat{C}, & a_1a_1a_1a_2a_2a_3/b_1b_2\hat{C}C\} \\ \{b_1b_1b_1b_1b_2b_2b_3/a_1\hat{C}, & b_1b_1b_1b_2b_2b_3/a_1a_2\hat{C}C\} \\ \{\hat{C}\hat{C}\hat{C}CC/a_1a_1a_2b_1b_1b_2, & \hat{C}\hat{C}\hat{C}\hat{C}CCc/a_1b_1\} \end{array}\right\} = j'''.$$

Define elements of $E^\times$ as follows, with $i, j = 1, 2, 3$:

$$\begin{aligned} W_{ij} &= a_ib_j\hat{C} \\ X_{ij} &= (a_1a_2a_3/a_i)(b_1b_2b_3/b_j)\hat{C}C \\ Y &= a_1a_2a_3b_1b_2b_3\hat{C}^2C \\ Z &= a_1a_2a_3b_1b_2b_3\hat{C}C^2 \end{aligned}$$

Then $j \in \alpha_6$ is plausible iff all nine W_{ij} are different from 1. $j' \in \alpha_5$ is plausible iff W_{22}, W_{23}, W_{32}, W_{33}, X_{22}, X_{23}, X_{32}, and X_{33} are different from 1. $j'' \in B_4$ is plausible iff W_{23}, W_{32}, X_{13}, X_{31}, X_{22}, X_{11}, and Y are different from 1. Finally $j''' \in H_2$ is plausible, or equivalently realizable, iff W_{33}, X_{12}, X_{21}, and Z are different from 1. Here we are using several times that $a_1^2a_2^2a_3^2b_1^2b_2^2b_3^2\hat{C}^3C^2c = 1$, as well as the plausibility condition (3.5) repeatedly. Altogether, one gets that j is realizable iff all nine W_{ij}, all nine X_{ij}, Y and Z are different from 1, as stated on the table. ∎

One can reinterpret Proposition 6.1 by thinking of A, a, b_1, b_2, … not as elements of $E^\times$, but rather as coordinate functions on a torus T covering J_{α_6}. Then Proposition 6.1 gives defining equations for the degeneracy locus $T^1 \subset T$ corresponding to non-realizable triples. An attribute of the proof is that it does not exploit the fact that T^1 is stable under the natural action of $S_3 \times S_3 \times S_1$ on T. Rather, computations such as (6.1) break such symmetries, since they involve arbitrary choices of markings. For example, in the case of α_6, the nine divisors (X_{ij}) are permuted transitively by $S_3 \times S_3 \times S_1$. But in (6.1), four of these divisors appear at the j' level, three more at the j'' level, and the final two at the j''' level. It would be desirable to be able to formulate Katz's classification in non-inductive terms, preferably in a way that was fully invariant under the action of these Weyl groups.

Remarks. The dual of a Jordan class $j_i \in J_n$ is obtained by replacing each eigenvalue by its inverse. Call a realizable rigid Jordan tuple $j = \{j_1, \ldots, j_z\}$ strictly self-dual if each j_i is self-dual. This notion plays a fundamental role in computing monodromy groups, for example. Table 6.1 interacts in interesting ways with this notion; for example, many of the families do not have strictly self-dual members for E of characteristic 2 as all singletons are forced to be 1, often contradicting plausibility.

7. Maximal series: H and P

The Hypergeometric series. For positive integers n, define a partition triple

$$H_n = \left\{ \begin{array}{c} n-1, 1 \\ 1, \dots, 1 \\ 1, \dots, 1 \end{array} \right\}.$$

Each H_n is rigid and plausible. H_1 is realizable. The unique derivative of H_n is H_{n-1}. So each H_n is realizable by induction.

The general member of the family J_{H_n} has the form

$$j = \left\{ \begin{array}{l} A^{n-1}, a \\ b_1, \cdots, b_n \\ c_1, \cdots, c_n \end{array} \right\}. \tag{7.1}$$

Here we regard all variables but a as freely chosen from $E^\times$. The variable a is then determined by the determinant condition (1.1). For $n > 1$, the Jordan triple j is plausible iff $Ab_ic_j \neq 1$ for all $i, j \in \{1, \dots, n\}$. With respect to the marking (A, b_n, c_n), the derivative of j is

$$\begin{array}{c} A \\ b_n \\ c_n \end{array} \left\{ \begin{array}{ll} (1/b_nc_n)^{n-2}, & a \\ (1/Ac_n)^0, & b_1, \dots, b_{n-1} \\ (1/Ab_n)^0, & c_1, \dots, c_{n-1} \end{array} \right\} \sim \begin{array}{c} Ab_nc_n \\ 1 \\ 1 \end{array} \left\{ \begin{array}{l} (1/b_nc_n)^{n-2}, a \\ b_1, \dots, b_{n-1} \\ c_1, \dots, c_{n-1} \end{array} \right\}$$
$$= \left\{ \begin{array}{l} A^{n-2}a' \\ b_1 \dots b_{n-1} \\ c_1 \dots c_{n-1} \end{array} \right\}$$

with $a' = Aab_nc_n$. Here $\sim$ indicates twisting by the rank one triple $(b_nc_n, b_n^{-1}, c_n^{-1})$, which changes neither plausibility or realizability. By induction, if j is plausible then j is realizable, in this classical case.

The Pochhammer series. For positive integers n, define a partition $(n+1)$-tuple

$$P_n = \{(n-1)1, \dots, (n-1)1\}.$$

Each P_n is rigid and plausible, with $P_2 = H_2$ and $P_1 = H_1$. The derivatives of P_n are

$$\partial_{(n-1,\dots,n-1)} P_n = H_1 \tag{7.2}$$
$$\partial_{(n-1,\dots,n-1,1)} P_n = P_{n-1}. \tag{7.3}$$

So realizability of P_n follows either directly from (7.2) or by induction from (7.3).

The general member of the family J_{P_n} has the form

$$j = \left\{ \begin{array}{c} A_1^{n-1}a_1 \\ \vdots \\ A_{n+1}^{n-1}a_{n+1} \end{array} \right\} \tag{7.4}$$

subject to Condition (1.1). The Jordan tuple j is plausible iff

$$\pi := A_1 \cdots A_{n+1} \neq 1 \tag{7.5}$$
$$\pi a_i / A_i \neq 1 \quad (i = 1, \dots, n+1) \tag{7.6}$$

both hold. One can check, in two ways, that here too plausibility implies realizability.

Remarks. There is an enormous literature on mathematics related to H_n or P_n. Particularly motivic references are [6] in the hypergeometric case and [2] in the Pochhammer case. A recent paper presenting H_n and P_n in tandem in the context of covers of the projective line is [11].

In general, given a realizable rigid Jordan tuple $j = \{j_1, \ldots, j_z\}$ one can ask for a matrix tuple $g = (g_1, \ldots, g_z) \in V(j_1, \ldots, j_z)$ representing it. In the hypergeometric and Pochhammer cases this problem has a uniform solution (see e.g. [11]). We restate the solution for H_n here in our language. By twisting, one can take $A = 1$ in (7.1). Define

$$\begin{aligned} f(x) &= \det(x - j_2) = (x - b_1)\cdots(x - b_n) = x^n + f_1 x^{n-1} + \cdots + f_{n-1}x + f_n \\ h(x) &= \det(x - j_3) = (x - c_1)\cdots(x - c_n) = x^n + h_1 x^{n-1} + \cdots + h_{n-1}x + h_n \end{aligned}$$

Define $e_0 = 1/(f_n h_n)$ and $e_i = h_i/(f_n h_n) - f_{n-i}/f_n$ for $i = 1, \ldots, n-1$. Define matrices by following the pattern evident in the case $n = 4$:

(7.7)
$$g_1 = \begin{pmatrix} e_0 & 0 & 0 & 0 \\ e_1 & 1 & 0 & 0 \\ e_2 & 0 & 1 & 0 \\ e_3 & 0 & 0 & 1 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 & 0 & 0 & -f_4 \\ 1 & 0 & 0 & -f_3 \\ 0 & 1 & 0 & -f_2 \\ 0 & 0 & 1 & -f_1 \end{pmatrix} \quad g_3 = \begin{pmatrix} -h_1 & 1 & 0 & 0 \\ -h_2 & 0 & 1 & 0 \\ -h_3 & 0 & 0 & 1 \\ -h_4 & 0 & 0 & 0 \end{pmatrix}.$$

Then $g_1 g_2 g_3 = 1$. The group $\langle g_1, g_2, g_3 \rangle$ acts irreducibly on E^n iff no root of f is the inverse of a root of h.

8. Submaximal series: A, B, C, D, E, F, G and I, J, K, L, M, N

In this section and the next, we work only at the partition level. Thus in these sections, we are only classifying realizable families, not examining the internal structure of a given family.

Call a realizable rigid partition tuple $\{\lambda_1, \ldots, \lambda_z\} \in rRPT$ *submaximal* if its length ℓ and its rank n satisfy the inequality $n + 5 \leq \ell < 2n + 2$. In this section, we classify submaximal partition tuples, restricting to $n \geq 6$ to avoid degeneracies.

On Table 8.1 we give twelve partition tuples in each $rRPT_{n,n+5}$. In each block the top line refers to n even and the bottom line refers to n odd. Thus our notation places these tuples in eleven full series and two half series G_{even} and L_{odd}.

For typographical reasons, we use the following abbreviations on Table 8.1:

$$\begin{aligned} u &= n/2 + 3/2 & \qquad v &= n/2 + 1 \\ w &= n/2 + 1/2 & x &= n/2 \\ y &= n/2 - 1/2 & z &= n/2 - 1. \end{aligned}$$

For n even only the integers v, x, and z appear; for n odd only the integers u, w, and y appear. Thus, to get a first understanding of Table 8.1, one can think in terms of u, v, w, x, y, $z \approx n/2$.

Note that series A, B, and C share a joint partition (3.2) and so do series D, E, F, and G:

$$(8.1) \qquad \mu = \begin{cases} xxxz1\cdots1 & \text{Series } ABC,\ n \text{ even} \\ wyyy1\cdots1 & \text{Series } ABC,\ n \text{ odd} \\ (n-2)2\cdots2111111 & \text{Series } DEFG,\ n \text{ even} \\ (n-2)2\cdots211111 & \text{Series } DEF,\ n \text{ odd.} \end{cases}$$

TABLE 8.1. The twelve tuples in $rRPT_{n,n+5}$ and their derivatives, $n > 6$

$z = 3$

	λ_1	λ_2	λ_3	1	2	> 2
A_n	xx	$xz1$	$1\cdots 1$	A		
	wy	$yy1$	$1\cdots 1$	A		
B_n	$xz1$	$x1\cdots 1$	$x1\cdots 1$	B, C		H_x, H_v
	$yy1$	$w1\cdots 1$	$y1\cdots 1$	B, C		H_x
C_n	xx	$x1\cdots 1$	$z1\cdots 1$	C		H_v
	wy	$y1\cdots 1$	$y1\cdots 1$	C		H_w, H_y
D_n	$(n-2)2$	$2\cdots 2$	$2\cdots 2111111$	E	D	
	$(n-2)2$	$2\cdots 21$	$2\cdots 211111$	D, E	D	
E_n	$(n-2)2$	$2\cdots 211$	$2\cdots 21111$	D, E	E	
	$(n-2)2$	$2\cdots 2111$	$2\cdots 2111$	E	E	
F_n	$(n-2)11$	$2\cdots 2$	$2\cdots 21111$	F	F	
	$(n-2)11$	$2\cdots 21$	$2\cdots 2111$	F, G	F	
G_n	$(n-2)11$	$2\cdots 211$	$2\cdots 211$	F	G	

$z = 4$

	λ_1	λ_2	λ_3	λ_4	1	2	> 2
I_n	$(n-1)1$	xx	$x1\cdots 1$	$v1\cdots 1$	I		H_v
	$(n-1)1$	wy	$w1\cdots 1$	$w1\cdots 1$	I		H_w
J_n	$(n-1)1$	$(n-1)1$	$2\cdots 2$	$2\cdots 211$	J	J	
	$(n-1)1$	$(n-1)1$	$2\cdots 21$	$2\cdots 21$	J	J	

$z \geq 5$ $\quad z = \lceil n/2 \rceil + 1$ (L, N) $\quad z = \lceil n/2 \rceil + 2$ (K, M)

	$\lambda_1, \lambda_4, \ldots$	λ_2	λ_3	1	2	> 2
K_n	$(n-1)1$	$xz1$	xx	K		P_v, P_x
	$(n-1)1$	wy	wy			P_u, P_w, P_y
L_n						
	$(n-1)1$	$yy1$	$yy1$	K		P_w

	$\lambda_1, \lambda_5, \ldots$	λ_2	λ_3	λ_4	1	2	> 2
M_n	$(n-2)2$	$(n-2)11$	$(n-1)1$	$(n-1)1$	M	M	H_2
	$(n-2)2$	$(n-1)1$	$(n-1)1$	$(n-1)1$	M	M	H_2
N_n	$(n-2)2$	$(n-2)11$	$(n-2)11$	$(n-2)11$	N	N	H_2
	$(n-2)2$	$(n-1)1$	$(n-2)11$	$(n-2)11$	N	N	H_2

The rigidity condition (1.2) is quickly verified in every case. For example, in Series ABC, n even, one has

$$\begin{aligned} 3x^2 + z^2 + (n+1)1^2 &= 3(\frac{n}{2})^2 + (\frac{n}{2} - 1)^2 + (n+1) \\ &= \frac{3}{4}n^2 + (\frac{1}{4}n^2 - n + 1) + (n+1) \\ &= n^2 + 2. \end{aligned}$$

Similarly plausibility condition (3.4) is quickly verified in all cases.

Table 8.1 gives all possible derivatives $Y_m \in RPT_m$ of each $X_n \in rRPT_n$. Here those with $m = n-1$ are placed in Column 1, those with $m = n-2$ are placed in Column 2, and any that remain are placed in Column >2. For example, consider B_n for n even; the four blocks of (8.2) account for the four derivatives of B_n printed on Table 8.1.

$$\begin{array}{llllll}
B_n = & \underline{x}z1 & \underline{x}11\cdots1 & x\underline{1}1\cdots1 & \\
 & zz1 & z11\cdots1 & x\ \ 1\cdots1 & = B_{n-1} \\
 & & & & \\
B_n = & xz\underline{1} & \underline{x}11\cdots1 & \underline{x}11\cdots1 & \\
 & xz & z11\cdots1 & z11\cdots1 & = C_{n-1} \\
 & & & & \\
B_n = & \underline{x}z1 & \underline{x}11\cdots1 & \underline{x}11\cdots1 & \\
 & \ \ z1 & \ \ 11\cdots1 & \ \ 11\cdots1 & = H_x \\
 & & & & \\
B_n = & x\underline{z}1 & \underline{x}11\cdots1 & \underline{x}11\cdots1 & \\
 & x\ 1 & 111\cdots1 & 111\cdots1 & = H_{x+1}.
\end{array} \tag{8.2}$$

The fact that our notation places the tuples in eleven full series and two half series has some artificial aspects; for example, F_{odd} is just as tightly tied to G_{even} as it is to F_{even}. On the other hand, the unions A, BC, DE, FG, I, J, KL, M, and N are stable under drop-1 and drop-2 derivation.

Theorem 8.1. *For $n \geq 6$, the set $rRPT_{n,n+5}$ contains exactly the twelve tuples listed in Table 8.1. Also $rRPT_{n,\ell}$ is empty for $n+5 < \ell < 2n+2$.*

Proof. Let

$$rRPT_{n,*} = \coprod_{\ell=n+5}^{2n+2} rRPT_{n,\ell}.$$

One can check, without using any classification results, that $rRJT_{n,*}$ is closed under derivation. One then has to check that if one starts with H_1 and successively antidifferentiates, keeping only realizable families with $\ell \geq n+5$, one gets only the families listed in Table 8.1. Here is a sample of the computations needed.

Consider antiderivatives λ of H_2 of rank $m+2$. They must consist of a copies of $m11$, $(3-a)$ copies of $(m+1)1$, and x copies of $m2$, for some $0 \leq a \leq 3$ and $x \in \mathbf{Z}_{\geq 0}$. The rigidity condition (1.2) for λ forces

$$a(m^2+1^2+1^2) + (3-a)((m+1)^2+1^2) + x(m^2+2^2) = (x+1)(m+2)^2 + 2$$

which simplifies to $2m(2x-m+a-1) = 0$. For $a = 0,1,2,3$, this gives M_{odd}, M_{even}, N_{odd}, and N_{even}, respectively, and nothing more. ∎

Remarks. Okubo, Yokoyama, Haraoka and others have studied the differential equations indexed by $\lambda = \{\lambda_1, \ldots, \lambda_z\} \in rRPT$ such that all the λ_i but perhaps one have the form $(m_i, 1, \ldots, 1)$. They prove that the possibilities are as follows:

	$z = 3$		$z > 3$	
n	Our notation	Their Notation	Our notation	Their Notation
Arbitrary	H_n	I_n	P_n	I_n^*
Even (≥ 4)	B_n	II_n	I_n	II_n^*
Odd (≥ 5)	B_n	III_n	I_n	III_n^*
6	F_6	IV_6	N_6	IV_6^*

See [3] for emphasis on differential equations; see [4] and [5] for explicit matrix formulas analogous to (7.7).

Simpson has approached our topic from a Hodge-theoretic viewpoint [10]. In our language, one of his classification results is the following. The only partition tuples $\lambda = \{\lambda_1, \ldots, \lambda_z\} \in rRPT$ with some $\lambda_i = 11 \cdots 11$ are the tuples H_n, A_n, and D_6.

9. Minimal series: α, β, γ, and δ

Call a partition tuple $\{\lambda_1, \ldots, \lambda_z\} \in RPT$ *minimal* if the corresponding length tuple $\{\ell_1, \ldots, \ell_z\}$ satisfies

$$\chi(\ell_1, \ldots, \ell_z) := \frac{1}{\ell_1} + \cdots + \frac{1}{\ell_z} - (z-2) \geq 0. \tag{9.1}$$

In this section, we classify minimal rigid partition tuples.

For $n \geq k$ positive integers, define $\mu_{n/k}$ to be the unique partition of n with k parts and all parts $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$. Thus $\mu_{n/k}$ is the "most balanced" partition of n into k parts. As three examples of this notation, γ_{13} below is $\{\mu_{13/2}, \mu_{13/3}, \mu_{13/6}\} = \{76, 544, 322222\}$. When k divides n, define also

$$\overline{\mu}_{n/k} := \{n/k+1, n/k, \cdots n/k, n/k-1\}.$$

In this case, $\mu_{n/k}$ is perfectly balanced and $\overline{\mu}_{n/k}$ is the next most balanced partition. As more examples of this notation, $\gamma_{12,2}$ below is $\{\overline{\mu}_{12,2}, \mu_{12/3}, \mu_{12/6}\} = \{75, 444, 222222\}$.

Theorem 9.1. *Put $\delta_k(n) = 1$ if k divides n and $\delta_k(n) = 0$ else. For $n > 6$, the set $RPT_{n,\min}$ contains exactly the $4 + \delta_4(n) + 2\delta_6(n)$ elements listed in Table 9.1. All these rigid partition tuples are realizable.*

Proof. The last statement follows by induction from the table of derivatives given. As to the completeness of the table, in general let $\lambda = \{\lambda_1, \ldots, \lambda_z\} \in RPT_n$ with corresponding length tuple $\{\ell_1, \ldots, \ell_z\}$. For each i, one has $n^2/\ell_i \leq ||\lambda_i||$ with equality iff ℓ_i divides n and $\lambda_i = \mu_{n/\ell_i}$. So one has

$$(z-2)n^2 + \chi(\ell_1, \ldots, \ell_z)n^2 \leq \sum_i ||\lambda_i|| = (z-2)n^2 + 2; \tag{9.2}$$

here the inequality is the definition of χ while the equality is Condition (1.2).

Suppose λ is minimal, meaning that $\chi(\ell_1, \ldots, \ell_z) \geq 0$. Then (9.2) is extremely restrictive and one can conclude by elementary arguments: the cases with $\chi > 0$ yield nothing with $n > 6$; the remaining cases $\{\ell_1, \ldots, \ell_z\} = \{3, 3, 3\}$, $\{2, 4, 4\}$, $\{2, 3, 6\}$, and $\{2, 2, 2, 2\}$ yield the series α, β, γ, and δ respectively. ■

TABLE 9.1. The $4 + \delta_4(n) + 2\delta_6(n)$ tuples in $RPT_{n,\min}$ and their derivatives

			Derivatives	
d	$n \in \mathbf{Z}/d$	Partition tuples	1	2
3	0	$\alpha_n = \{\overline{\mu}_{n/3}, \mu_{n/3}, \mu_{n/3}\}$	α_{n-1}	
	1	$\alpha_n = \{\mu_{n/3}, \mu_{n/3}, \mu_{n/3}\}$	α_{n-1}	α_{n-2}
	2	$\alpha_n = \{\mu_{n/3}, \mu_{n/3}, \mu_{n/3}\}$	α_{n-1}	
4	0	$\beta_{n,2} = \{\overline{\mu}_{n/2}, \mu_{n/4}, \mu_{n/4}\}$	β_{n-1}	
	0	$\beta_{n,4} = \{\mu_{n/2}, \overline{\mu}_{n/4}, \mu_{n/4}\}$	β_{n-1}	
	1	$\beta_n = \{\mu_{n/2}, \mu_{n/4}, \mu_{n/4}\}$	$\beta_{n-1,2},\ \beta_{n-1,4}$	β_{n-2}
	$2, 3$	$\beta_n = \{\mu_{n/2}, \mu_{n/4}, \mu_{n/4}\}$	β_{n-1}	
6	0	$\gamma_{n,2} = \{\overline{\mu}_{n/2}, \mu_{n/3}, \mu_{n/6}\}$	γ_{n-1}	
	0	$\gamma_{n,3} = \{\mu_{n/2}, \overline{\mu}_{n/3}, \mu_{n/6}\}$	γ_{n-1}	
	0	$\gamma_{n,6} = \{\mu_{n/2}, \mu_{n/3}, \overline{\mu}_{n/6}\}$	γ_{n-1}	
	1	$\gamma_n = \{\mu_{n/2}, \mu_{n/3}, \mu_{n/6}\}$	$\gamma_{n-1,2},\ \gamma_{n-1,3},\ \gamma_{n-1,6}$	γ_{n-2}
	$2, 3, 4, 5$	$\gamma_n = \{\mu_{n/2}, \mu_{n/3}, \mu_{n/6}\}$	γ_{n-1}	
2	0	$\delta_n = \{\overline{\mu}_{n/2}, \mu_{n/2}, \mu_{n/2}, \mu_{n/2}\}$	δ_{n-1}	
	1	$\delta_n = \{\mu_{n/2}, \mu_{n/2}, \mu_{n/2}, \mu_{n/2}\}$	δ_{n-1}	δ_{n-2}

The four series just discussed are naturally associated with the four rank two Dynkin diagrams: $(\alpha, \beta, \gamma, \delta) \leftrightarrow (A_2, B_2, G_2, A_1 \times A_1)$; this connection partially explains our choice of notation.

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