A hypergeometric exploration of the geography of pure motives

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(reporting on ongoing work with
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Structure of the talk

Our aim is to computationally explore the category $\mathcal{M} = \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ of pure motives over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$. An irreducible motive M of weight w and rank n has a Sato-Tate group

$$G \subseteq Sp_n \text{ (odd w)}, \qquad G \subseteq O_n \text{ (even w)}.$$

We call M *full* if equality holds dimensionally: $G \in \{Sp_n, O_n, SO_n\}$.

1. Hypergeometric motives (HGMs)
2. Two closely related general inverse problems:

Hodge IP: (Pursued in an unconditional context) What Hodge vectors $h = (h^{-w}, h^{2-w}, \ldots, h^{w-2}, h^w)$ can arise from full motives?

Minimal conductor problem : (Pursued assuming general expectations about conductors) For a given realizable h, what's the smallest conductor N_h of a full motive with Hodge vector h?

3. Affirmative answers to the Hodge IP for many h from HGMs
4. Upper bounds on some N_h from HGMs

Section 1

Hypergeometric Motives

Formalism of hypergeometric motives (HGMs)

Let n be a positive integer. Let $f(x), g(x) \in \mathbb{Z}[x]$ be coprime monic polynomials of degree n with all roots being roots of unity. Then for any $t \in \mathbb{Q} - \{0, 1\}$ one has a corresponding rank n motive $H(f(x), g(x), t) \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$.

Example with $n = 6$:

$$\begin{aligned} f(x) &= \Phi_2(x)^2\Phi_8(x) = (x+1)^2(x^4+1), \\ g(x) &= \Phi_3(x)^2\Phi_6(x) = (x^2+x+1)^2(x^2-x+1), \\ t &= 4/3. \end{aligned}$$

Magma allows many computations with HGMs. For example:

```
H := HypergeometricData([2,2,8],[3,3,6]);
L := LSeries(H,4/3);
EulerFactor(L,7);
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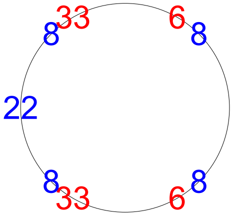
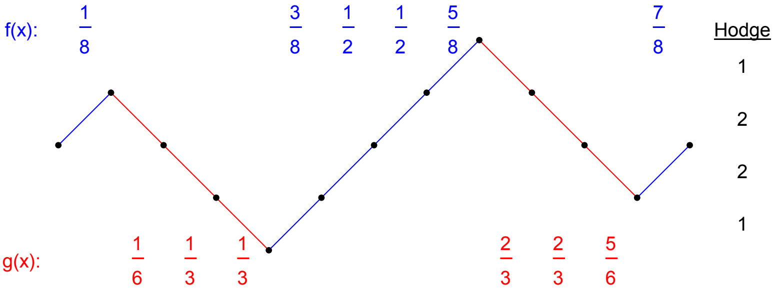
$1 + 12x - 2\cdot 7^2x^2 - 59\cdot 7^2x^3 - 2\cdot 7^5x^4 + 12\cdot 7^6x^5 + 7^9x^6$

Hodge vectors of HGMs

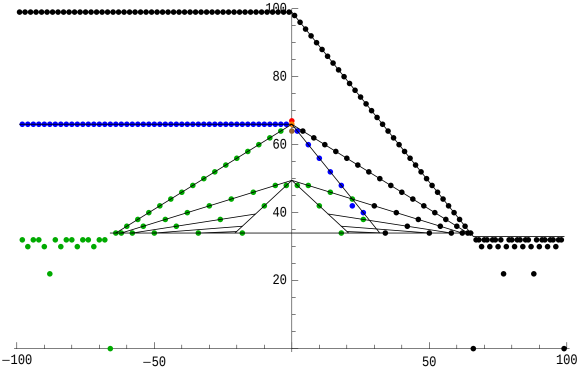
The Hodge vector of an HGM is independent of the specialization point t. It is calculated by how roots intertwine on the unit circle, as illustrated by $f(x) = \Phi_2(x)^2\Phi_8(x)$ and $g(x) = \Phi_3(x)^2\Phi_6(x)$:





Conductors of HGMs: an example

Consider $H([66,22,6,2],[33,11,3,1],t)$ which has Hodge vector (33) so its conductor $N(t)$ can be computed as a ratio of number field discriminants. Write $t = 2^k u$. The numbers $c_2(t) = \mathrm{ord}_2(N(t))$ are given by the following functions of k:



Black=Any u. Blue: $u \equiv 3\ (4)$. $u \equiv 1\ (4)$.

Conductors of HGMs

Motivated by many pictures very analogous to the previous one, we have an elaborate but incomplete conjectural theory of ramification in HGMs. Its most basic part for a fixed prime p is as follows. Define (Artin slopes)

$$\alpha_p(j) = \begin{cases} 1 & \text{if } j = 0 \quad \text{(tame ramification)}, \\ 1 + j + \frac{1}{p-1} & \text{if } j > 0 \quad \text{(wild ramification)}. \end{cases}$$

For a hypergeometric family $H(A, B)$ define

$$\sigma_\infty = \sum_{a \in A} \phi(a) \alpha_p(\mathrm{ord}_p(a)), \qquad \sigma_0 = \sum_{b \in B} \phi(b) \alpha_p(\mathrm{ord}_p(b)).$$

Define also $\delta = \sigma_\infty - \sigma_0$.

Conductors of HGMs, continued

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be the unique continuous piecewise linear function with corners only above 0 and δ and satisfying

$$\sigma(k) = \begin{cases} \sigma_\infty & \text{if } k \leq \min(0, \delta), \\ \sigma_0 & \text{if } k \geq \max(0, \delta). \end{cases}$$

We also define a slightly smaller function $\underline{\sigma}$ by looking more closely at tame ramification.

Let $c(t)$ be $\mathrm{ord}_p(\mathrm{Conductor}(H(A, B, t)))$, so that c is a locally constant function on $\mathbb{Q}_p - \{0, 1\}$. We conjecture

$$c(p^k u) \leq \underline{\sigma}(k).$$

Moreover, we expect equality to hold if and only if $\mathrm{ord}_p(k) = 0$ or $\sigma(k) = \mathrm{rank}(H(A, B))$.

Section 2

Context on the inverse Hodge and minimal conductor problems

Motives from modular forms with trivial character

Let w be an odd positive integer. Then isomorphism classes of motives $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ with Hodge vector

$$(1, \overbrace{0, \ldots, 0}^{w-1}, 1)$$

should be in bijection with normalized Hecke newforms f of modular weight $w + 1$ on $\Gamma_0(N)$ with rational coefficients.

Non-full $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ correspond to rational forms f with complex multiplication. They exist for all w.

Full $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ correspond to rational forms f without complex multiplication. They exist for $w = 1, 3, 5, 7, \ldots, 43, 45, 47, 49$, and I conjecture they do not exist for $w \geq 51$. Minimal conductors are

w	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49
N	11	5	3	2	2	1	2	1	1	1	1	2	1	2	2	2	2	2	6	2	2	6	6	2	6

Four types of h

h is called *rigid* if it has an interior zero and *mobile* else. From an algebro-geometric perspective, it should be harder to find motives for rigid h, because (outside the exceptional O_4 case) Griffiths transversality prevents them from moving in families.

Examples, all symplectic with rank four:

	mobile	rigid
regular	$(1,1,1,1)$	$(1,0,0,1,0,0,0,0,0,0,1,0,0,1)$
irregular	$(2,2)$	$(2,0,0,0,0,0,0,2)$

h is called *regular* if all its entries are 0's or 1's, except perhaps for a central 2. It is called *irregular* else. It is harder to use the automorphic approach to find motives in $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ for irregular h, because they can't be isolated by the trace formula.

General expectations on inverse Hodge problem

Our best guesses, with each extra "?" indicating greater doubt.

	mobile	rigid
regular	$(1,1,1,1,1,2,1,1,1,1,1)$ **Yes** by HGMs	1011010001000020000100010101 **Usually no?** Like rank two case
irregular	$(2,1,1,4,1,1,2)$ **Always yes??** See Prop. below	$(3,2,1,0,0,1,2,3)$ **????** see some examples below

For the four rank twelve examples, the answer is yes.

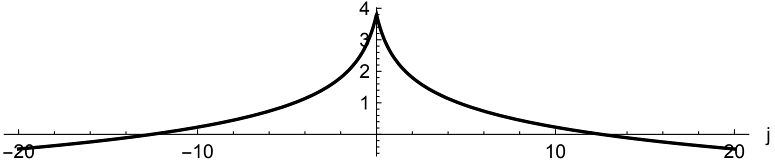
(Much recent work has been done on the rigid regular case, with many examples with $N = 1$.)

Rough expectations on minimal conductors

Let

$$\Lambda(j) = \log(2\pi) - \psi(\frac{|j|+1}{2}).$$

Its graph, with asymptotic approximant $\log(4\pi/|j|)$, is



One has $\Lambda(0) = \log(8\pi e^{\gamma}) \approx \log(44.76) \approx 3.80$. One can view

$$A_h = \exp\left(\sum_j h^j \Lambda(j)\right)$$

as an "analytic first guess" (needing considerable further refinement!) at the minimal conductor N_h.

Section 3

Affirmative solutions to the inverse Hodge problem for many h from HGMs

A. Mobile Hodge vectors in ranks ≤ 24

Proposition. *In ranks ≤ 24, every mobile Hodge vector h comes from a full hypergeometric motive, except the following twelve Hodge vectors, all orthogonal:*

<u>Rank 20</u>

$(6, 1, 1, 1, 2, 1, 1, 1, 6)$

<u>Rank 22</u>

$(6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 6)$
$(4, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 4)$

<u>Rank 23</u>

$(1, 21, 1)$

<u>Rank 24</u>

$(9, 1, 1, 2, 1, 1, 9)$
$(7, 1, 1, 1, 1, 2, 1, 1, 1, 1, 7)$
$(6, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 6)$
$(5, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 5)$
$(4, 1, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 4)$
$(1, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 6, 1)$
$(4, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 4)$
$(4, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 4)$

A. Ranks ≤ 24, continued

A family $H(f(x), g(x))$ is called *primitive* if $f(x)/g(x)$ is not a function of x^k for some $k \geq 2$. The Zariski closure of the monodromy group of the primitive family $H(f(x), g(x))$ is the entire symplectic or orthogonal group whenever $w \geq 1$. This implies $H(f(x), g(x), t)$ is full for "almost all" specialization points t.

The proposition is then proved by direct computation. For example, there are 319,685,444 symplectic families in rank 24, but only $2^{11} = 2048$ Hodge vectors. The average fiber size is thus

$$r_{24} = \frac{319685444}{2048} \approx 156096.$$

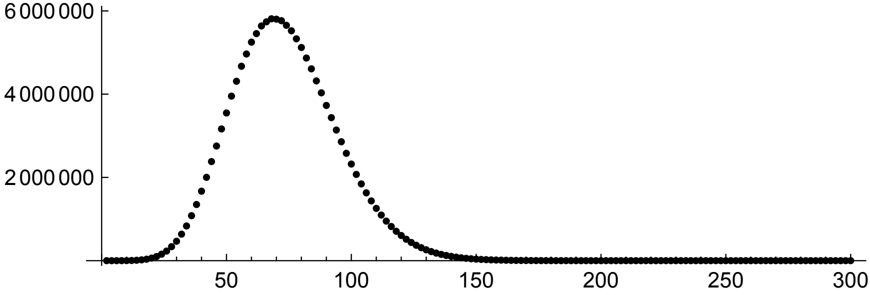
Thus it is not surprising that all fibers are non-empty, as asserted by the proposition. In fact, the smallest fibers occur above

$$(5, 1, 2, 1, 1, 2, 2, 1, 1, 2, 1, 5) \text{ and } (8, 1, 1, 2, 2, 1, 1, 8)$$

and have size 34.

A. Larger ranks

The average number of symplectic rank n families per Hodge vector is r_n, graphed as follows.



The maximum ratio is $r_{68} \approx 5,810,819$. The last data point is $r_{300} \approx 0.000013$. So in large ranks, hypergeometric motives answer the Hodge inverse problem positively only for a vanishingly small fraction of mobile Hodge vectors.

B. Rigid solutions from HGMs at $t = 1$

One can also specialize hypergeometric families at the mild singular point $t = 1$.

All Hodge numbers stay the same except:

- When w is even, the central Hodge number drops by 1, as in $(2, 3, 1, 3, 2) \to (2, 3, 0, 3, 2)$.
- When w is odd, the two centermost Hodge numbers drop by 1, as in $(2, 3, 1, 1, 3, 2) \to (2, 3, 0, 0, 3, 2)$.

Fullness fails in the *reflexive* case $f(x) = (-1)^n g(-x)$, because of an operator inherited from $t \mapsto 1/t$.

Outside the $w = 0$, the imprimitive, and the reflexive cases, it seems likely that fullness holds for all but finitely many $(f(x), g(x))$. Fullness can be verified for given $(f(x), g(x))$ by computing two sufficiently different Euler factors, as on the next slide.

B. Rigid solutions from HGMs at $t = 1$, continued

Example:

```
H:=HypergeometricData([3,3,3,3],[1,1,1,1,1,1,1,1]);
L:=LSeries(H,1);
```

The Hodge vector is $(1,1,1,0,0,1,1,1)$ by the above procedure.

```
f2 := EulerFactor(L,2); f2;
```

$1 + 9x + 39 \cdot 2x^2 + 207 \cdot 2^3x^3 + 39 \cdot 2^8x^4 + 9 \cdot 2^{14}x^5 + 2^{21}x^6$

```
f5 := EulerFactor(L,5); f5;
```

$1 + 18x - 4416 \cdot 5x^2 + 65592 \cdot 5^3x^3 - 4416 \cdot 5^8x^4 + 18 \cdot 5^{14}x^5 + 5^{21}x^6$

Both polynomials are conformally palindromic sextics, so their Galois group is within $\mathrm{Weyl}(Sp_6) = 2^3 : S_3$ of order 48. The biggest their joint Galois group could be is $48 \cdot 48 = 2304$. Indeed:

```
Order(GaloisGroup(f2*f5));
```

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This suffices to show that $H([3,3,3,3],[1,1,1,1,1,1,1,1],1)$ is full.

C. Rigid solutions from reflexive HGMs

Reflexive motives $H(f(x), -f(-x), (-1)^n)$ decompose as the sum of two motives in $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$ of equal or near-equal ranks.

Example. For the rank 14 motive $M = H([2^{16}], [1^{16}], 1)$, the decomposition on Hodge vectors is

$$\begin{aligned}(1,1,1,1,1,1,1,0,0,1,1,1,1,1,1,1) &= \\ (1,0,1,0,1,0,1,0,0,1,0,1,0,1,0,1) &+ \\ (0,1,0,1,0,1,0,0,0,0,1,0,1,0,1,0)\end{aligned}$$

For $p = 3$, 5, the Euler factor $\det(1 - x\mathrm{Fr}_p|M)$ factors into an irreducible sextic and an irreducible octic. The nature of the irreducible factors confirms both fullness and the Hodge numbers.

(The rank six factor has conductor 2^6 reflecting tame ramification only. the rank eight factor has conductor 2^9.)

C. Rigid solutions from reflexive HGMs

In general, we conjecture that the decomposition $h = h_1 + h_2$ has the "maximal fairness" property:

The numbers $h_1^{p,q} - h_2^{p,q}$ are all in $\{-1, 0, 1\}$ with the non-zero differences alternating in sign for $p \geq q$.

We also conjecture fullness of each summand outside of a finite number of exceptions.

This would give positive solutions to the Hodge inverse problem for infinitely many difficult-looking h. For example, we'd expect that the two summands of

$$H([27, 2^{16}], [54, 1^{16}], 1)$$

are both full, with rigid irregular Hodge vectors

$$(3, 2, 1, 0, 0, 0, 0, 1, 2, 3) \text{ and } (2, 3, 0, 1, 0, 0, 1, 0, 3, 2).$$

Section 4

4. Upper bounds on N_h for many h from HGMs

A. Basal HGMs

Every family $H(A, B, t)$ has two preferred specialization points

$$t = \pm \prod p^{\delta_p}.$$

These "basal points" are the unique points which are at the bottom corner of the ramp for all p. We conjecture that

$$N(t) \;\text{ divides }\; \mathrm{red}(\mathrm{denom}(t) - \mathrm{num}(t)) \prod_p p^{\min(\sigma_{\infty,p}, \sigma_{0,p}) - n}.$$

We expect equality often, including always in the common case that no prime is involved in both A and B, so that the product is 1. Here

$$\mathrm{red}(s) = \begin{cases} \mathrm{radical}(s) & \text{if } w \text{ is odd,} \\ \mathrm{squarefreepart}(s) & \text{if } w \text{ is even.} \end{cases}$$

A. Basal HGMs, two examples

A conductor drop at one specialization point can be exploited in infinitely many examples. For example, take $t = -1/1024$ so that

$$\mathrm{denom}(t) - \mathrm{num}(t) = 1024 + 1 = 1025 = 41 \cdot 5^2.$$

Indexing by rank, let

$$\begin{aligned} M_4 &= H([4,2,2],[1,1,1,1],-1/1024), \\ M_5 &= H([2,2,2,2,2],[1,1,1,1,1],-1/1024). \end{aligned}$$

Hodge vectors and conductors are

$$\begin{aligned} &h_4 = (1,1,1,1), &&N_4 = 41 \cdot 5 = 205, &&(N_{(1,1,1,1)} = 61) \\ &h_5 = (1,1,1,1,1), &&N_5 = 41. \end{aligned}$$

The motive M_4 is full and has been matched with a Siegel modular form. In contrast, M_5 is not full. It has a full submotive with $h = (1,1,0,1,1)$; this submotive corresponds to a Hilbert modular form on $\mathbb{Q}(\sqrt{41})$ and weight $(2,4)$.

B. Special HGMs

Specializations at $t = 1$ tend to have smaller conductors. E.g. for $H([4, 2^{k-2}], [1^k], 1)$, conductors are 2^c with

k	3	4	5	6	7	8	9	10	11	12	13	14
n	2	2	4	4	6	6	8	8	10	10	12	12
c	3	**4**	9	**8**	9	**12**	13	**16**	*11*	**20**	19	**24**

For basal special HGMs, such as $H([2^{3a+8b}], [4^a, 8^b, 1^{a+4b}], 1)$, conductors are even smaller:

Ranks n

$b \setminus a$	0	1	2	3	4	5	6
0		2	4	8	10	14	16
1	6	10					
2	14						

Conductor Exponents c

$b \setminus a$	0	1	2	3	4	5	6
0		3	**5**	11	**12**	18	**19**
1	**7**	*12*					
2	**18**						

In the red example, the Hodge vector is $h = (1, 1, 1, 0, 0, 1, 1, 1)$ and the conductor is $2^7 = 128$. The analytic guess at the minimal conductor is $A_h \approx 338.3$.

C. Semi HGMs

Recall that each reflexive index (A, B) gives a decomposing motive.

Example: $H([3,2,2,2,2],[6,1,1,1,1],1) = M_{1,0,0,1} \oplus M_{1,0,0,0,0,1}$.

A second reason that these semi HGMs are interesting is that they are automatically lightly ramified. At 2, their ramification can be studied by comparing with a dyadic HGM, here $H([2,2],[1,1],1)$. At odd primes p, the ramp has length zero, making making them basal.

The modular forms for the example are $f \in S_4(12)$ and $\eta_2^{12} \in S_6(4)$. These are the ninth and second smallest conductors for their Hodge vector.

Similarly, $H([3,2^6],[6,1^6],1) = M_{1,0,0,0,0,1} \oplus M_{1,0,1,0,0,1,0,1}$ with $M_{1,0,0,0,0,1}$ from $\eta_2^{12} \in S_6(4)$ again and $M_{1,0,1,0,0,1,0,1}$ of conductor $2^5 3 = 96$.

Selected References

The talk is presently being converted to a paper.

Varieties underlying hypergeometric motives are described in Frits Beukers, Henri Cohen, and Anton Mellit, *Finite hypergeometric functions*, Pure Appl. Math. Q. 11 (2015), no. 4, 559–589.

The formula for Frobenius traces is proved there, and is closely related to earlier formulas of Greene and Katz.

The *Magma* hypergeometric package is due to Mark Watkins. The *Magma* L-function package used to calculate and/or confirm conductors is due to Tim Dokchitster.

The Hodge number formula is proved in Roman Fedorov. *Variations of Hodge structures for hypergeometric differential operators and parabolic Higgs bundles*, Int. Math. Res. Notices, 2017. Antecedents include works of Terasoma, Corti, Golyshev, Dettweiler, and Sabbah.

Selected References, continued

For motives with Hodge vector $(1, 0, \ldots, 0, 1)$, see my *Newforms with rational coefficients*, Ramanujan J., 2018.

The sample degree twelve realizable regular rigid Hodge vector was drawn from Olivier Taïbi, *Dimensions of spaces of level one automorphic forms for split classical groups using the trace formula.* Ann. Sci. Éc. Norm. Supér. (4) 50 (2017), no. 2, 269–344. Related works include Chenevier-Renard, Chenevier-Lannes, and a website by Bergström, Faber, and van der Geer.

Lower bounds for conductors in a motivic context were given by J.-F. Mestre, *Formules explicites et minorations de conducteurs de variétés algébriques.* Comp. Math. 58 (1986), no. 2, 209–232.

Similar lower bounds obtained by focusing on the tensor square L-function were given by Gaëten Chenevier, *An automorphic generalization of the Hermite-Minkowski theorem*, ArXiv, 2018