Hypergeometric Belyi Maps

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Overview

Classical hypergeometric functions with rational parameters

$$ {}_nF_{n-1}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_{n-1}, t) $$

have monodromy matrices g_∞, g_1, $g_0 \in GL_n(\mathbb{Q}^{\mathrm{cyclo}})$ satisfying

$$ g_\infty g_1 g_0 = 1. $$

Beukers and Heckman (1989) classified the exceptional case when the image $G = \langle g_\infty, g_0 \rangle$ is finite.

One can also take images in $GL_n(\overline{\mathbb{F}}_\ell)$ where they are always finite. Then one gets an *infinite hierarchy* of *combinatorially indexed*, *tightly interrelated*, *arithmetically important* Belyi maps. Today is an introduction to this hierarchy. I will show several examples of hypergeometric Belyi maps. Our main example has image $Sp_8(\mathbb{F}_2)$ and goes beyond the Beukers-Heckman list.

Combinatorial indexing

The classical theory sets $\beta_n = 1$. This is a bad convention! It is better to work with a general β_n. The arithmetically most important case is α_j, $\beta_j \in \mathbb{Q}$. Monodromy depends only on $\alpha_j, \beta_j \in \mathbb{Q}/\mathbb{Z}$. Define

$$q_\infty(x) = \prod_{j=1}^{n} \left(x - e^{2\pi i \alpha_j}\right). \qquad q_0(x) = \prod_{j=1}^{n} \left(x - e^{-2\pi i \beta_j}\right).$$

We simplify by requiring $q_\infty(x), q_0(x) \in \mathbb{Z}[x]$ and coprime. Integrality allows *cyclotomic indexing* where e.g. $\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$ is written as [5].

Main example. Take $(\alpha, \beta) =$

$$\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}; \ \frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{7}{15}, \frac{8}{15}, \frac{11}{15}, \frac{13}{15}, \frac{14}{15}\right) = ([3, 9], [15]).$$

Then

$$q_\infty(x) = \Phi_3(x)\Phi_9(x), \qquad q_0(x) = \Phi_{15}(x).$$

Monodromy matrices (Levelt 1961)

Let A and B be the standard companion matrices of $q_\infty(x)$ and $q_0(x)$. Then

$$(g_\infty, g_1, g_0) := (A, A^{-1}B, B^{-1}).$$

Main example of $([3,9],[15])$ **again.** The polynomials are

$$q_\infty(x) = x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, \quad q_0(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1.$$

The matrix product $g_\infty g_1 g_0$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here g_∞, g_1, and g_0 (obviously!) have orders 9, ∞, and 15.

Examples with monodromy S_c from $a + b = c$

An easy and famous collection of examples comes from

$$q_\infty(x) = \frac{x^c - 1}{x - 1}, \qquad q_0(x) = \frac{(x^a - 1)(x^b - 1)}{x - 1}.$$

Here $c = n + 1 = a + b$ with a and b coprime. The monodromy is S_c and an equation for a Belyi map $\mathbb{P}^1_x \to \mathbb{P}^1_t$ is

$$c^c x^a (1 - x)^b - t a^a b^b = 0.$$

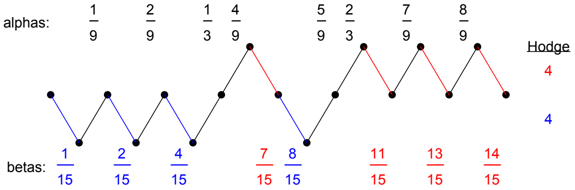
When $b = 1$ the equation is trinomial. Always, the equation can be rewritten as $y^c + uy^a + v = 0$ with u and v monomials in t.

The Grothendieck-Beckmann theorem allows bad reduction at all $p \leq c$. But bad reduction is only at $p | abc$. This good feature is shared by all hypergeometric Belyi maps.

Varieties and their Hodge vectors

The local system with monodromy (g_∞, g_1, g_0) comes from a family of varieties X_t degenerating only at ∞, 1 and 0 [BCM]. The Hodge numbers $h^{p,q}$ of the main piece in the middle cohomology can be computed by how the α and β intertwine.

Procedure illustrated by the **main example of** $([3,9],[15])$:



Here the Hodge vector is $h = (h^{1,0}, h^{0,1}) = (4,4)$, which matches the fact that the X_t are genus four curves. The extreme possibilities are

No intertwining:	$h = (1, \dots, 1)$	(of Calabi-Yau interest),
Complete intertwining:	$h = (n)$	(Finite monodromy [BH]).

The Beukers-Heckman list

Beukers and Heckman classified the 76 exceptional cases with finite monodromy, allowing the coefficients of $q_\infty(x)$ and $q_0(x)$ to generate an arbitrary number field $E \subset \mathbb{Q}^{\mathrm{cyclo}}$. The six last groups on the Shephard-Todd list of complex reflection groups from 1954 account for two-thirds of the Beukers-Heckman list:

E	Name	G			BH	N
$\mathbb{Q}(\sqrt{-3})$	Maschke	$ST32$	$=$	$3 \times Sp_4(\mathbb{F}_3)$	24-36	27
$\mathbb{Q}(\sqrt{-3})$	Burkhardt	$ST33$	$=$	$SO_5(\mathbb{F}_3)$	41-44	27
$\mathbb{Q}(\sqrt{-3})$	Mitchell	$ST34$	$=$	$3.SO_6^-(\mathbb{F}_3)$	50-57	112
$\mathbb{Q}$	"27 lines"	$ST35$	$=$	$W(E_6) = SO_6^-(\mathbb{F}_2)$	45-49	27
$\mathbb{Q}$	"28 bitangents"	$ST36$	$=$	$W(E_7) = 2 \times Sp_6(\mathbb{F}_2)$	58-62	28
$\mathbb{Q}$	"120 tritangents"	$ST37$	$=$	$W(E_8) = 2.SO_8^+(\mathbb{F}_2)$	63-77	120

N is the minimal degree of the Belyi map corresponding to the projective representation. To get the linear representation one may need a larger degree.

Equations for Beukers-Heckman covers

In the "27 lines" and "28 bitangents" cases, covering curves always have genus zero. Some of the equations $tA(x) + B(x) = 0$ correspond to 1990s papers, others are also easy from a modern viewpoint:

BH	α's	β's	$A(x)$	$B(x)$
45	$[3,12]$	$[1,2,8]$	$2^4x^3\left(x^2-3\right)^{12}$	$-3^9(x-2)(x-1)^8\left(x^2-2x-1\right)^8$
46	$[3,12]$	$[1,2,5]$	$5^5(x-1)^3\left(2x^2+2x-1\right)^{12}$	$-2^{10}3^9x^5(2x-1)^2\left(x^2+x-1\right)^5$
47	$[9]$	$[1,2,4,6]$	$\left(x^3+6x^2-8\right)^9$	$-2^43^{12}(x-2)x^6(x+1)^4(x+4)^4$
48	$[9]$	$[1,2,8]$	$2^{18}\left(x^3+9x^2+6x+1\right)^9$	$-3^{15}x(2x+1)^8\left(x^2-2x-1\right)^8$
49	$[9]$	$[1,2,5]$	$2^25^5\left(x^3-3x^2+1\right)^9$	$-3^{15}(x-1)^2x^5\left(x^2-x-1\right)^5$
58	$[2,18]$	$[1,3,12]$	$-2^{18}\left(x^3+3x^2-3\right)^9$	$3^6x^3(3x+4)\left(x^2+6x+6\right)^{12}$
59	$[2,18]$	$[1,3,5]$	$5^5(x-1)\left(x^3+9x^2+15x-1\right)^9$	$2^{14}3^{12}x^3\left(x^2+7x+1\right)^5$
60	$[2,18]$	$[1,7]$	$7^7\left(8x^3+36x^2+12x+1\right)^9$	$2^{14}3^{15}x^7\left(x^3-20x^2-9x-1\right)^7$
61	$[2,14]$	$[1,3,12]$	$2^{18}3^9\left(x^3-7x+7\right)^7$	$7^7(x-1)(x+3)^3\left(x^2-3\right)^{12}$
62	$[2,14]$	$[1,3,5]$	$3^35^5(x+1)^7\left(x^3-x^2-9x+1\right)^7$	$2^{14}7^7x^3\left(x^2+3x+1\right)^5$

In other cases, covering curves almost always have higher genus [R]. I first obtained equations for them by specializing generic polynomials for complex reflection groups. This process is sometimes easy, sometimes hard, and I am still missing two Mitchell cases.

Mod ℓ representations

A hypergeometric datum (α, β) and a prime ℓ determine a Belyi map capturing $(g_\infty, g_1, g_0) \in GL_n(\mathbb{F}_\ell)^3$. For small degrees n and primes $\ell \in \{2, 3\}$, the Belyi maps are often on the Beukers-Heckman list.

Example. Consider three hypergeometric data (using cyclotomic abbreviation as in $[8] = \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$ again):

α's	β's	Hodge vector	Nature of varieties X_t
[8]	[1, 1, 1, 1]	(1, 1, 1, 1)	Threefold
[8]	[3, 1, 1]	(2, 2)	$y^2 = x^5 + 4tx^2 + 3tx$
[8]	[3, 3]	(2, 2)	Threefold

The indices agree up to 3-torsion so the three (g_∞, g_1, g_0) are exactly the same in $GL_4(\mathbb{F}_3)^3$. In fact this mod 3 representation comes from BH28 with $(\alpha, \beta) = \left(\frac{1}{24}, \frac{7}{24}, \frac{13}{24}, \frac{19}{24};\ 0, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}\right)$. In fact, all 13 Maschke cases BH24-BH36 likewise come from $y^2 = x^5 + ax^3 + bx^2 + cx + d$.

The main example: A hypergeometric Belyi map with monodromy group $Sp_8(\mathbb{F}_2)$

Step 1. Start from $([3,9],[15])$ and $\ell = 2$.

Step 2. Writing $s = \frac{5^{10}}{2^4 3^6} t$, the genus four curves X_t are given by

$$F(x,y) = \begin{array}{cccc} 1 & & & +sy^3 \\ & & -3sxy^2 & \\ & -3sx^2y & & -s^2x^2y^3 \\ -sx^3 & & & \end{array} = 0.$$

For $t \neq 0,\ 1,\ \infty$ they are smooth curves of bidegree $(3,3)$ in the ambient space $\mathbb{P}^1_x \times \mathbb{P}^1_y$.

Step 3. A generic curve $G(x,y) = xy + ax + by + c = 0$ of bidegree $(1,1)$ meets X_t in six distinct points, with x-values the roots of $\text{Resultant}_y(F(x,y), G(x,y)) = 0$. Require that this sextic have the form $-s(\text{cubic})^2$ to get three explicit equations on a, b, c.

Main example of ([3,9],[15]) with $\ell = 2$, continued

Step 4. Remove c by a resultant (easy). Remove b by a resultant (harder; one minute on *Magma*). Clean up by writing $a = \frac{2^2 3^3}{5^5} x$.

The final polynomial has 213 terms:

$$\beta(t,x) = s^{56} x^{120} + 2^9 s^{55} x^{117} + \cdots + 2^{16} 3^{12} 5^{10}.$$

It has ramification triple $(\lambda_\infty, \lambda_1, \lambda_0) = (9^{13}3, 2^{28}1^{64}, 15^8)$. Its singular specialization is $\beta(1,x) = f_{28}(x)^2 f_{64}(x)$ with $f_{28}(x) =$

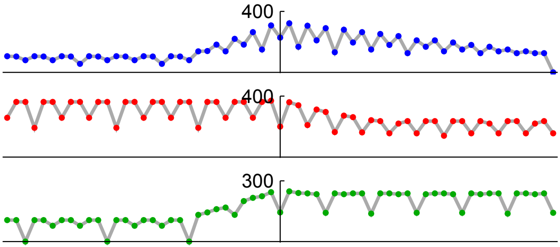
$$\begin{aligned}
&729x^{28} + 14580x^{27} - 729000x^{25} - 364500x^{24} + 18808200x^{23} - 10813500x^{22} \\
&- 246523500x^{21} + 332606250x^{20} + 2032020000x^{19} - 4588582500x^{18} - 8854312500x^{17} \\
&+ 32255043750x^{16} + 13751437500x^{15} - 123395906250x^{14} + 1415250000x^{13} \\
&+ 395654765625x^{12} - 230352187500x^{11} - 807579687500x^{10} + 1036859375000x^{9} \\
&+ 283591406250x^{8} - 326203125000x^{7} - 1762746093750x^{6} + 529453125000x^{5} \\
&+ 6155009765625x^{4} - 9439570312500x^{3} + 3423339843750x^{2} + 2734375000000x \\
&- 1938232421875.
\end{aligned}$$

$$\mathrm{Gal}(f_{28}(x)) = Sp_6(\mathbb{F}_2) \text{ and } \mathrm{fielddisc}(f_{28}(x)) = 2^{42} 3^{52} 5^{30}.$$

Belyi maps as a tool for studying wild ramification

Wild ramification at p in varieties over $\mathbb{Q}$ is always faithfully represented in mod ℓ representations for any $\ell \neq p$. For hypergeometric varieties, wild ramification is mysterious. Explicit equations for mod ℓ representations are a useful tool.

Graphs of $\mathrm{ord}_p(\text{fielddisc}(\beta(-p^k, x)))$ for $k \in [-30, 30]$ and $p = 2,3,5$:



It would be hard to get this information about ramification in hypergeometric varieties in other ways. It is useful for example for determining conductors of L-functions.

Selected references

[BH] Frits Beukers and Gert Heckman. *Monodromy for the hypergeometric function* ${}_nF_{n-1}$. Invent. Math. 95 (1989), no. 2, 325–354.

[BCM] Frits Beukers, Henri Cohen, Anton Mellit. *Finite hypergeometric functions*, Pure Appl. Math. Q. 11:4 (2015), 559–589.

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[ST] G. C. Shephard and J. A. Todd (1954), *Finite unitary reflection groups*, Can. J. Math., 6 (1954), 274–304.