Modularity problems for hypergeometric motives

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First half: Context-Setting Background
Second half: Semi Hypergeometric Motives

[One year math position available.
Email me at roberts@morris.umn.edu
if you or someone you know is interested.]

Hodge vectors

For a genus g curve X, the Hodge numbers of $H^1(X, \mathbb{Q})$ are $(h^{1,0}, h^{0,1}) = (g, g)$.

For a K3 surface X, there is a decomposition

$$H^2(X, \mathbb{Q}) = H^2_{\mathrm{trans}}(X, \mathbb{Q}) \oplus H^2_{\mathrm{alg}}(X, \mathbb{Q}).$$

On Hodge vectors $(h^{2,0}, h^{1,1}, h^{0,2})$ the decomposition is

$$(1, 20, 1) = (1, a, 1) + (0, 20 - a, 0)$$

for some a in $\{0, \ldots, 19\}$.

In general a weight w *Hodge vector* is a vector $h = (h^{w,0}, \ldots, h^{0,w})$ of non-negative integers satisfying the Hodge symmetry $h^{p,q} = h^{q,p}$ and the normalization condition $h^{w,0} \geq 1$. Its *rank* is $n = \sum_{p=0}^{w} h^{p,w-p}$.

Seeking motives for a given $h = (h^{w,0}, \ldots, h^{0,w})$

Restrict to $w \geq 1$ as $w = 0$ behaves differently. Mentally focus on unusual Hodge vectors like $(3, 1, 0, 0, 0, 0, 0, 0, 1, 3)$.

Precise theoretical Q1. Does there exist a full motive M having Hodge vector h?

["Full" means "with motivic Galois group as large as possible" and can be viewed as a nondegeneracy condition. We'll answer *yes* for many h just by exhibiting full motives. I conjecture that the answer is *no* for many h.]

Vague computational Q2. For a motive M as found for Q1, can we numerically verify that its L-function $L(M, s)$ has the expected analytic properties? Can we numerically find an automorphic representation π with $L(\pi, s) = L(M, s)$?

Tables in rank two and odd weight w

For $h = (1, \overbrace{0, 0, \ldots, 0, 0}^{w-1}, 1)$, the motives sought are, under standard expectations, in canonical bijection with newforms of modular weight $k = w + 1$ with rational coefficients and without CM.

The LMFDB provides excellent tables. E.g., for $h = (1, 0, 0, 1)$ fill the input boxes with

Conductor = Level = $\boxed{\text{1..100}}$ Modular weight $k = \boxed{4}$

Self-Twists = $\boxed{\text{no CM}}$ Coefficient Field = $\boxed{1.1.1.1}$

The list of 168 forms is complete, conductors being $N = 5, 6, 7, \ldots$

In particular the answer to Question 1 is *yes* for $w = 1, 3, 5, \ldots, 49$. I've conjectured that it's *no* for $w = 51, 53, \ldots$. The reason is that all newspaces have dimension ≥ 2, and exceptions to a general Maeda principle that newforms in a given newspace are conjugate have only been seen for $w \leq 25$.

Hypergeometric Motives

Let $A = [A_1, \ldots, A_r]$ and $B = [B_1, \ldots, B_s]$ be multisets of positive integers with $A_i \neq B_j$ always and $n = \sum_i \phi(A_i) = \sum_j \phi(B_j)$. The pair (A, B) determines a family of varieties X_t degenerating only at $t \in \{0, 1, \infty\}$. For example, take $(A, B) = ([3, 4], [5])$. Then X_t is the genus two curve determined by $2^6 3^3 t(xy^2 + 1) = 5^5 x^2(1 + xy)$.

The rank n hypergeometric motive $H(A, B; t)$ sits in the cohomology of X_t. Its Hodge vector is determined by how the roots of $\Phi_{A_1}(T) \cdots \Phi_{A_r}(T)$ and $\Phi_{B_1}(T) \cdots \Phi_{B_r}(T)$ intertwine in the unit circle. Complete intertwining gives $h = (n)$ and complete separation gives $(1, 1, \ldots, 1, 1)$. *In general, these h from HGMs never have 0's.*

So for many Hodge vectors without 0's, e.g. all of them in ranks ≤ 21, one gets a positive answer to Question 1. Fullness is easily proved via monodromy. Exotic Hodge vectors like $(9, 1, 1, 1, 1, 1, 1, 9)$ can arise from thousands of families, each with infinitely many specialization points!

Hypergeometric L-Series

For $t \in \mathbb{Q} - \{0, 1\}$, *Magma* computes all the good factors of $L(H(A, B; t), s)$, and makes often-correct guesses at the bad primes:

```
H := HypergeometricData([3,4],[5]);
L := LSeries(H,25/24:Precision:=7);
Conductor(L);
```

$1\,822\,500\,000 = 2^5 3^6 5^7$

```
LCfRequired(L);
```

222 111 terms needed at this precision

```
CFENew(L);
```

0.000000 (7 minutes) All looks correct!

```
[Evaluate(L,1), Evaluate(L,1:Derivative:=1)]
```

$[0.000000, 7.840885]$ (6 minutes) Critical vanishing

To better respond to Question 2, we need to keep conductors very small for the given h! An easy way is to involve just one small prime in (A, B). A quite different way with interesting special features is the focus of the second half of the talk!

Parameters for semiHGMs

Trivially $H(A,B;t)$ is isomorphic to $H(B,A;1/t)$. We can't take $A=B$ because of our non-degeneracy condition. But we can require

$$\begin{array}{rcl} A & = & A_{\text{odd}} \cup 2B_{\text{odd}} \\ B & = & 2A_{\text{odd}} \cup B_{\text{odd}} \end{array} \quad \text{as in} \quad \begin{array}{rcl} A & = & [1,1,3,10] \\ B & = & [2,2,6,5]. \end{array}$$

For (A,B) of this very special form, there is a decomposition

$$H(A,B;(-1)^n) =: H(A_{\text{odd}},B_{\text{odd}}) = H^-(A_{\text{odd}},B_{\text{odd}}) \oplus H^+(A_{\text{odd}},B_{\text{odd}}).$$

Ranks of the semiHGMs are

				n^-	n^+
Orthogonal case	n	$=$	$2r+1:$	r	$r+1$
Balanced symplectic case	n	$=$	$4g+2:$	$2g$	$2g$
Unbalanced symplectic case	n	$=$	$4g+4:$	$2g$	$2g+2$

For $H([1,1,3],[5])$, the ranks are $n^-=2$ and $n^+=4$. Despite the sin of involving three primes, the conductor $N^-=5$ is first on the $(1,0,0,1)$ list and $N^+=150$ is fairly early on the $(1,1,1,1)$ list.

Hodge vectors of semiHGMs

Computation with examples strongly suggests that Hodge vectors of semiHGMs are universally determined by a "maximal fairness" property: *The numbers* $h_+^{p,q} - h_-^{p,q}$ *are in* $\{-1, 0, 1\}$ *with nonzero differences alternating in sign for* $p \geq q$.

For $H([1^{10}7], [\,])$ with $(n^-, n^+) = (6, 8)$ the prediction is

$$\begin{aligned} h &= (4, 1, 1, 1, 0, 0, 1, 1, 1, 4) \\ h_- &= (2, 0, 1, 0, 0, 0, 0, 1, 0, 2) \\ h_+ &= (2, 1, 0, 1, 0, 0, 1, 0, 1, 2) \end{aligned}$$

It can be proved in instances arithmetically using the close relation between Hodge numbers and p-adic ordinals of roots of Frobenius polynomials $f_p(x) = f_p^-(x) f_p^+(x)$. In our example,

$$f_{\mathbf{5}}^-(x) = (1 + 5^{\mathbf{27}} x^{\mathbf{6}}) - 2601(x + 5^{\mathbf{18}} x^{\mathbf{5}}) + 5032463(x^{\mathbf{2}} + 5^{\mathbf{9}} x^{\mathbf{4}}) - 258824158 \cdot 5^{\mathbf{2}} x^{\mathbf{3}}$$

$$f_{\mathbf{5}}^+(x) = (1 + 5^{\mathbf{36}} x^{\mathbf{8}}) + 659(x + 5^{\mathbf{27}} x^{\mathbf{7}}) + 1602654(x^{\mathbf{2}} + 5^{\mathbf{18}} x^{\mathbf{6}}) + 742226797(5x^{\mathbf{3}} + 5^{\mathbf{10}} x^{\mathbf{5}}) + 3751560002 \cdot 5^{\mathbf{4}} x^{\mathbf{4}}$$

The partition of the 5-adic ordinals given by the two irreducible factors proves maximal fairness.

Fullness and Question 1

Computation with examples strongly suggests that semiHGMs are almost always full. Continuing with the example $H([1^{10}7],[\,])$,

$$f_3^-(x) = (1+3^{27}x^6) - 55(x+3^{18}x^5) - 1535(x^2+3^9x^4) + 132914\cdot 3^3x^3$$
$$f_3^+(x) = (1+3^{36}x^8) - 25(x+3^{27}x^7) + 560(x^2+3^{18}x^6) + 173701(3x^3+3^{10}x^5) - 3819794\cdot 3^4x^4$$

Letting $W(C_r) = 2^r S_r$ be the group of r-by-r signed permutation matrices,

$$\mathrm{Gal}(f_3^-(x)f_5^-(x)) = W(C_3)\times W(C_3),$$
$$\mathrm{Gal}(f_3^+(x)f_5^+(x)) = W(C_4)\times W(C_4).$$

The fact that these Galois groups are as large as possible suffices to prove fullness of $H^-([1^{10}7],[\,])$ and $H^+([1^{10}7],[\,])$.

Unlike HGMs at $t\neq 1$ from the first half of the talk, semiHGMs can answer Question 1 with *yes* for h which contain 0's. This is a qualitative improvement: motives with such Hodge vectors cannot move in families and so only can be found one-by-one.

Bad reduction of semiHGMs

Reduction at bad primes is very structured. Sample expectations:

For p = 2, write $n_a = \sum_{A_i \in A_{\mathrm{odd}}} \phi(A_i)$ and $n_b = \sum_{B_i \in B_{\mathrm{odd}}} \phi(B_i)$ so that $n = n_a + n_b$. Then 2-adic ramification generally increases with $|n_a - n_b|$. Two extremes:

Condition	$\mathrm{ord}_2(N^-)$	$\mathrm{ord}_2(N^+)$	$f_2^-(x)$	$f_2^+(x)$
$n_a = n_b$	0	1	$g_2(x)$	(lin)$g_2(x)$
$0 \in \{n_a, n_b\}$, n odd	$n-1$	$n+2$	1	1

Here (lin) is a specified linear factor and $g_2(x)$ is a specified Tate twist of the Frobenius polynomial for $H(A_{\mathrm{odd}}, B_{\mathrm{odd}}; 1)$.

For odd primes p, most and typically all of $f_p(x)$ is likewise obtained by "erasing" A_i and B_j whenever they are multiples of p. Ramification is tame at p if and only if p^2 does not divide any of the A_i and B_j.

An orthogonal example of rank $4+5$

There are 35 cases with $(n, n^-, n^+) = (9, 4, 5)$. The only one ramified at 2 only is $H([1^9], [\,])$. It actually has only the fifth smallest conductor, because the ramification at 2 is relatively bad.
Information on the decomposition $H^-([1^9], [\,]) \oplus H^+([1^9], [\,])$:

h	N	form
$(1,0,1,0,1,0,1)$	2^8	$M_f \otimes M_g$ with $\begin{cases} f \in S_3(32, \chi_{-4}) \\ g \in S_5(32, \chi_{-4}) \end{cases}$
$(1,0,1,0,1,0,1,0,1)$	2^{11}	[challenge!]

Here the newforms are

$$\begin{aligned} f &= q - 4iq^3 + 2q^5 + 8iq^7 + \cdots \\ g &= q - 4iq^3 + 26q^5 - 88iq^7 + \cdots \end{aligned}$$

In the tensor product, the irrationality $i = \sqrt{-1}$ goes away.

A symplectic example of rank 4+6

There are 97 cases with $(n, n^-, n^+) = (12, 4, 6)$. All but four of them give two full motives.

One of the four exceptions is $H([1^{10}], [3])$ with Frobenius polynomials

$$
\begin{aligned}
f_5(x) &= (1 - 54 \cdot 5^3 x + 5^{11} x^2)(1 + 666 \cdot 5x + 5^{11} x^2)(\text{sextic}) \\
f_7(x) &= (1 + 88 \cdot 7^3 x + 7^{11} x^2)(1 + 904 \cdot 7x + 7^{11} x^2)(\text{sextic})
\end{aligned}
$$

The evidence is overwhelming that $H^-([1^{10}], [3])$ decomposes:

h	N	form	Placement
$(1,0,0,0,0,1)$	2^2	$q \prod_{k=1}^{\infty} (1 - q^2)^{12}$	second
$(1,0,0,0,0,0,0,0,0,1)$	2^2	$q + 228q^3 - 666q^5 \cdots$	fourth
$(1,0,1,0,1,0,0,1,0,1,0,1)$	$2^7 3$	[challenge!]	[challenge!]

Knowing $L(H^-([1^{10}], [3]), s)$ lets one compute $L(H^+([1^{10}], [3]), s)$ in isolation. E.g. $L(H^+([1^{10}], [3]), 6) \approx 1.1770515607675810065242197$ is calculated in half a minute.

Some references

The survey paper *Hypergeometric Motives* with Fernando Rodriguez Villegas will appear in the Notices of the AMS this summer. It points to many relevant points in the hypergeometric literature.

Newforms with rational coefficients in the Ramanujan Journal states and supports the conjecture about the nonexistence of such forms in high weights.

The L-function and HGM parts of *Magma* are absolutely essential to the computational exploration of HGMs and semiHGMs. They are due to Tim Dokchitser and Mark Watkins respectively.

A paper corresponding to this talk is in preparation. The above two papers and these slides are at `www.davidproberts.net`.