Hypergeometric Motives

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1 Introduction

It must have been frustrating in the early days of calculus that an integral like

$$F(t) = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{x(1-x)(1-tx)}}\,dx \qquad (1.1)$$

appeared not to be expressible in terms of known functions. This type of integral arises in computing the movement of the ideal pendulum or the length of an arc of an ellipse for example; they have remained relevant and are connected to a great deal of the mathematics of the last 200 years.

Indeed F is not an elementary function. Its Maclaurin expansion

$$F(t) = \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{t}{16}\right)^k \qquad (1.2)$$

is an example of a hypergeometric series. It satisfies a linear differential equation of order two of the type brilliantly analyzed by Riemann. As mentioned by Katz [Kat96, p.3], Riemann was lucky. His analysis only works because any order two differential equation on $\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}$ is *rigid* in the sense that the local behavior of solutions around the missing points uniquely determines their global behavior.

Taking a more geometric perspective, (1.1) is presenting the function πF as a period of the family of elliptic curves defined by

$$E_t: \quad y^2 = x(1-x)(x-t). \qquad (1.3)$$

This fact implies as well that F satisfies an order two linear differential equation, ultimately because for $t \neq 0,1$, the set $E_t(\mathbb{C})$ of complex points is a topological torus and so $H^1(E_t(\mathbb{C}),\mathbb{Q})$ is two-dimensional. The only cohomology we will encounter in this paper is classical singular cohomology; accordingly we abbreviate by $H^1(E_t,\mathbb{Q}) := H^1(E_t(\mathbb{C}),\mathbb{Q})$ and systematically omit similar references to $\mathbb{C}$ in the sequel.

Shifting now to more arithmetic topics, if we fix a rational number $t \neq 0,1$, then the numbers a_p defined by

$$|E_t(\mathbb{F}_p)| = p + 1 - a_p \qquad (1.4)$$

for almost all primes p are of fundamental importance. With these a_p as the main ingredients, one builds an L-function

$$L(E_t, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \qquad (1.5)$$

Much of the importance of the a_p is seen through this L-function. For example, the famous conjecture of Birch and Swinnerton-Dyer says that the group $E_t(\mathbb{Q})$ modulo its torsion is isomorphic to $\mathbb{Z}^r$, where r is the order of vanishing of $L(E_t,s)$ at $s=1$. A critical advance is the proof by Wiles *et al.* that the function

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i z} \qquad (1.6)$$

on the upper half plane is a modular form.

The discussion so far represents a standard general paradigm in arithmetic geometry. One starts with a smooth projective variety X of any dimension κ over $\mathbb{Q}$, not just a curve as in the family (1.3), and focuses on invariants associated to its cohomology groups $H^k(X,\mathbb{Q})$. Typically it is enough to consider the middle cohomology $k=\kappa$ as invariants of other cohomology groups can be studied using subvarieties of X. Among the theoretically well-understood invariants are periods like (1.1)–(1.2) and Frobenius traces like the a_p in (1.4). More mysterious is an L-function analogous to (1.5), expected to satisfy some well-known standard properties. More mysterious still is the conjectural analog of (1.6), which in general takes a much more complicated form. The space $H^\kappa(X,\mathbb{Q})$ is an example of a motive and a full understanding of the general paradigm requires the theory of motives.

This paper is an informal invitation to *hypergeometric motives*, hereafter called HGMs. They are very concrete so that their definition which we give in Section 4 can be understood with just the minimal background on motives that we provide there. We write an HGM as $H(\gamma,t)$. The *gamma vector* $\gamma = [\gamma_1,\ldots,\gamma_l]$ is any weakly increasing sequence of nonzero integers summing to zero, with no two γ_i being negatives of one another; the next section explains how it arises. The *specialization point* t is a rational number different from 0 and 1. The examples (1.1)–(1.6) fit into the hypergeometric framework via

$$H([-2,-2,1,1,1,1],t) = H^1(E_t,\mathbb{Q}). \qquad (1.7)$$

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Sections 2–9 are geometric in nature. The main focus is on varieties generalizing (1.3) and the discrete aspect of periods like (1.1)–(1.2), as captured in vectors of Hodge numbers, $h = (h^{w,0}, h^{w-1,1}, \ldots, h^{1,w-1}, h^{0,w})$. There can be different varieties X underlying a given $H(\gamma, t)$ and the Hodge vector h captures their most important common feature. For example, if one X is a genus g curve and the inclusion $H(\gamma, t) \subseteq H^1(X, \mathbb{Q})$ is equality then $h = (g, g)$.

Sections 10–15 are arithmetic in nature, with the focus being on generalizations of (1.4), (1.6), and especially on the L-functions $L(H(\gamma, t), s)$ generalizing (1.5). Watkins has written a very useful hypergeometric motive package [Wat15] in Magma and throughout this article we indicate how to use it by including small snippets of Magma code. Together these snippets are enough to let Magma beginners numerically compute with hypergeometric L-functions using the free online Magma calculator.

Rigidity makes HGMs much more tractable than general motives: periods, Frobenius traces, and other invariants are given by explicit formulas in the parameters (γ, t). What makes them really exciting is that at the same time, HGMs form a broad class of motives covering very general values of h.

2 Family parameters

We begin by generalizing (1.1)–(1.2) using classical parameters. We then describe how in the cases we are interested in the same parameter data can be encoded in two other formats: the γ vector already mentioned and a rational function Q. Each of these descriptions of the same data has its own advantages and we will freely use one or the other depending of the context.

Integrals and series. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $\beta = (\beta_1, \ldots, \beta_n)$ be vectors of complex numbers with $\mathrm{Re}(\beta_j) > \mathrm{Re}(\alpha_j) > 0$ and $\beta_n = 1$. For $|t| < 1$ define, making use of the standard Gamma function $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$,

$$F(\alpha, \beta, t) = \prod_{i=1}^{n} \frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i - \alpha_i)} \cdot \qquad (2.1)$$
$$\int_0^1 \cdots \int_0^1 \frac{\prod_{i=1}^{n-1} \left(x_i^{\alpha_i - 1} (1 - x_i)^{\beta_i - \alpha_i - 1} dx_i \right)}{(1 - t x_1 \cdots x_{n-1})^{\alpha_n}}.$$

Via $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$, (1.1) is the special case $\alpha = (1/2, 1/2)$ and $\beta = (1, 1)$.

Expanding the denominator of the integrand of (2.1) via the binomial theorem and using Euler's beta integral to evaluate the individual terms, one obtains

$$F(\alpha, \beta, t) = {}_nF_{n-1}(\alpha, \beta, t) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \ldots (\beta_n)_k} t^k, \qquad (2.2)$$

where $(u)_k = u(u+1)\cdots(u+k-1)$ is the Pochhammer symbol. In other words, the integral (2.1) is an alternative definition of the standard hypergeometric power series (2.2). The case $\alpha = (1/2, 1/2)$ and $\beta = (1, 1)$ simplifies to (1.2).

Monodromy. An excellent general reference for hypergeometric functions is [BH89], and we now give a brief summary. The function $F(\alpha, \beta, t)$ is in the kernel of an n^{th} order differential operator $D(\alpha, \beta)$ with singularities only at 0, 1 and ∞. This means in particular that $F(\alpha, \beta, t)$, initially defined on the unit disk, extends to a "multivalued function" on the thrice-punctured projective line $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\} = \mathbb{C} - \{0, 1\}$. With respect to a given basis, this multivaluedness is codified by a representation ρ of the fundamental group $\pi_1(\mathbb{C} - \{0, 1\}, 1/2)$ into $\mathrm{GL}_n(\mathbb{C})$. The fundamental group is free on g_0 and g_1, with these elements coming from counterclockwise circular paths of radius $1/2$ about 0 and 1 respectively. To emphasize the equal status of ∞ and 0, it is better to present this group as generated by g_∞, g_1, and g_0, subject to the relation $g_\infty g_1 g_0 = 1$. The assumption $\beta_n = 1$ was only imposed to present the classical viewpoint (2.1)-(2.2) cleanly; we henceforth drop it.

A useful fact due to Levelt is the explicit description of the matrices $h_\tau = \rho(g_\tau) \in \mathrm{GL}_n(\mathbb{C})$ with respect to a certain well-chosen basis. Define polynomials

$$q_\infty := (T - e^{2\pi i \alpha_1}) \cdots (T - e^{2\pi i \alpha_n}),$$
$$q_0 := (T - e^{-2\pi i \beta_1}) \cdots (T - e^{-2\pi i \beta_n}).$$

Then h_∞ and h_0 are companion matrices of q_∞ and q_0, while h_1 is determined by $h_\infty h_1 h_0 = I$. The matrix h_1 differs minimally from the identity in that $h_1 - I$ has rank 1. We will henceforth consider only cases where no $\alpha_i - \beta_j$ is an integer. This ensures that the h_τ generate an irreducible subgroup Γ of $GL_n(\mathbb{C})$. Moreover the representation is rigid, in the following sense: suppose h'_∞, h'_1, and h'_0 are conjugate to h_∞, h_1, and h_0 respectively. Then there is a single matrix c such that $h_\tau = c h'_\tau c^{-1}$ for all three τ.

Other formats. A way to encode the classical parameters α and β is to consider the rational funtion $Q := q_\infty / q_0$. An advantage is that redundancies are eliminated, as α_i and β_j only matter modulo integers and their ordering is irrelevant. By construction, all three h_τ lie in $GL_n(E)$, where E is the subfield of $\mathbb{C}$ generated by the coefficients of q_∞ and q_0. We require henceforth that all the α_i and β_j are rational, making E a subfield of a cyclotomic field. In this setting, an underlying motivic theory is presented in [Kat90, §5.4]. We substantially simplify by restricting to the cases with $E = \mathbb{Q}$ in this survey.

We call Q a *family parameter*; its degree is the common degree of q_∞ and q_0, i.e. the order of the differential operator $D(\alpha, \beta)$. Another equivalent way to encode the hypergeometric data is to write

$$Q(T) = \frac{\prod_{\gamma_i < 0} (T^{-\gamma_i} - 1)}{\prod_{\gamma_i > 0} (T^{\gamma_i} - 1)},$$

for non-zero integers γ_i with no two of them being negatives of one another. It is simple to see that this uniquely determines the γ_i up to reordering. Moreover, since q_∞ and q_0 have the same degree, the γ_i's sum to zero. In other words, standardizing $\gamma = [\gamma_1, \ldots, \gamma_l]$ to be weakly increasing, we have a gamma vector as defined in the introduction.

As an illustration the two formats for the parameter data in our introductory example are

$$\gamma = [-2,-2,1,1,1,1], \qquad Q = \frac{(T^2-1)^2}{(T-1)^4} = \frac{\Phi_2^2}{\Phi_1^2}, \tag{2.3}$$

where Φ_k is the k-th cyclotomic polynomial. The simplified form on the right hand side of the expression for Q has numerator q_∞ and denominator q_0.

To enter a family parameter Q into Magma, one can use either of these two ways, as in the equivalent commands

```
Q:=HypergeometricData([*-2,-2,1,1,1,1*]);
Q:=HypergeometricData([1,1],[2,2]);   (2.4)
```

In the first method, one inputs just the gamma vector. In the second method, one inputs just the subscripts of the denominator and then numerator Φ's. When working with underlying varieties, the gamma vectors are so useful that we typically denote HGMs by $H(\gamma, t)$, as in the introduction. After the transition to motives, the cyclotomic presentation is generally more convenient and we usually write $H(Q, t)$, giving Q in lowest terms. To simplify slightly, we henceforth require that $\gcd(\gamma_1, \ldots, \gamma_l) = 1$.

Three Magma notes. Initialization commands like (2.4) do not return output; in this survey, Magma will first start returning useful information in Sections 5 and 11. Also essential to know is that Magma requires a semicolon at the end of all commands, as in (2.4). We omit these semicolons in our in-line text when giving explicit Magma commands in the sequel. Finally, Magma conventions are reversed from the classical conventions, as α_i's are before β_j's in (2.2) but their abbreviations-by-denominator are written in the other order in (2.4). As another example of this reversal, $\alpha = (\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$ and $\beta = (\frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6})$ in (2.1)-(2.2) are input by `[4,6],[5]` in the analog of (2.4).

3 Source varieties

There are different choices of varieties that can be used to define hypergeometric motives. Here we use certain affine varieties $X_{\gamma,t}$ that we call *canonical*. They appear under the term "circuits" in [GKZ94] and are studied at greater length by Beukers, Cohen, and Mellit in [BCM15].

The canonical variety $X_{\gamma,t}$. For a gamma vector γ and a nonzero complex number t, define $X^{\mathrm{bcm}}_{\gamma,t} \subset \mathbb{P}^{l-1}$ by two homogeneous equations,

$$\sum_{j=1}^{l} z_j = 0, \qquad \prod_{\gamma_j>0} z_j^{\gamma_j} = u \prod_{\gamma_i<0} z_j^{-\gamma_j}. \tag{3.1}$$

Here and in the sequel, we systematically use the abbreviation $u = t\prod_j \gamma_j^{\gamma_j}$. The canonical variety from this viewpoint is then the open subvariety $X_{\gamma,t}$ of $X^{\mathrm{bcm}}_{\gamma,t}$ on which all the homogeneous coordinates z_j are nonzero. The point $(\gamma_1 : \cdots : \gamma_l)$ is an ordinary double point on $X_{\gamma,1}$ and otherwise all the $X_{\gamma,t}$ are smooth. Because of this double point, we exclude the case $t = 1$ from consideration until Section 9.

Toric models. The BCM equations (3.1) are appealing since they define $X_{\gamma,t}$ directly from the gamma vector γ, without any choices. However, it is typically more convenient to consider a toric model as in [GKZ94]. To obtain a such a model from γ, one proceeds as illustrated by Table 3.1. First, for $i = 1, \ldots, d$, choose a row vector m_{i*} in

$$\begin{array}{r} \\ x_1: \\ x_2: \\ u: \end{array} \begin{array}{|cccc|} \hline \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \hline m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ \hline k_1 & k_2 & k_3 & k_4 \\ \hline \end{array} = \begin{array}{|cccc|} \hline -5 & -2 & 3 & 4 \\ \hline 2 & 1 & 0 & 3 \\ 0 & 2 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline \end{array}$$

Table 3.1: Derivation of the equation (3.3) for $X_{[-5,-2,3,4],t}$

$\mathbb{Z}^l$ which is orthogonal to γ. Second, choose a row vector $k \in \mathbb{Z}^l$ which satisfies $\gamma \cdot k = 1$. The vectors m_{i*} together with k and $(1, 1, \ldots, 1, 1)$ are required to span $\mathbb{Z}^l$. The toric model is then

$$\sum_{i=1}^{l} u^{k_i} \prod_{j=1}^{d} x_i^{m_{ij}} = 0. \tag{3.2}$$

So in the example of Table 3.1, the resulting equation is

$$x_1^2 + ux_1x_2^2 + u + x_1^3x_2 = 0, \tag{3.3}$$

with $u = -2^6 3^3 t/5^5$. The curves for $t = 1/9$, 1, and 9 are drawn in the opening figure, with the ordinary double point at $(x_1, x_2) = (-24/25, 5/6)$ visible in the case $t = 1$. In general, the variety $X_{\gamma,t}$ is the subvariety of the torus $\mathbb{G}_m^d$ given by the equation (3.2).

Toric models as parameterizations. The relation between the BCM equations (3.1) and a toric model for the same variety $X_{\gamma,t}$ is simple:

$$z_j = u^{k_j} \prod_{i=1}^{d} x_i^{m_{ij}}. \tag{3.4}$$

When one uses (3.4) to write (3.1) in terms of the x_i, the second equation is identically satisfied while the first becomes (3.2). Conversely, any point $(z_1 : \cdots : z_l)$ comes from a unique $(x_1, \ldots, x_d)$ because of the spanning condition we imposed on our choice of m_{i*}.

Polytopes. A toric model determines a polytope $\Delta \subset \mathbb{R}^d$ which is an aid to understanding the $(d-1)$-dimensional variety $X_{\gamma,t}$. The case $d = 2$ is readily visualizable and Figure 3.1 continues the example of Table 3.1. In general, one interprets the column vectors m_{*j} of the chosen matrix as points in $\mathbb{Z}^d$ and Δ is their convex hull. Let Δ_j be the convex hull of all the points except the j^{th} one. Normalize volume so that the standard d-dimensional simplex has volume 1, and thus $[0,1]^d$ has volume $d!$. Then the volume of Δ_j is $|\gamma_j|$. The Δ_j with $\gamma_j > 0$ form one triangulation of Δ, while the Δ_j with $\gamma_j < 0$ form another. The total volume of Δ is the important number $\mathrm{vol}(\gamma) = \frac{1}{2}\sum_{j=1}^{l}|\gamma_j|$.

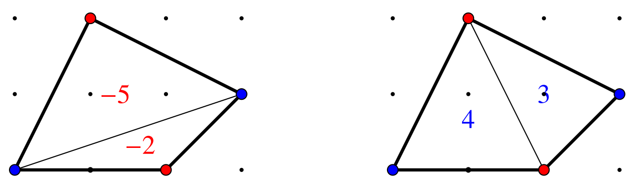

Figure 3.1: The triangulations $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta = \Delta_3 \cup \Delta_4$ of the polytope Δ for the family with $\gamma = [-5,-2,3,4]$. The invariant γ_j is printed in the triangle Δ_j opposite to the vertex $\binom{m_{1j}}{m_{2j}}$ from Table 3.1.

The common topology of the $X_{\gamma,t}$ with $t \neq 1$ is reflected in the combinatorics of Δ. In the case of curves, the numerical invariants of $X_{\gamma,t}$ are the genus g of its smooth compactification and the number k of points added to obtain this compactification. Very simply, g is the number of lattice points in the interior of Δ, while k is the number of lattice points on the boundary. Pick's theorem then says that the Euler characteristic $\chi = 2-2g-k$ of $X_{\gamma,t}$ is $-\mathrm{vol}(\gamma)$. In the example of Figure 3.1, $(g,k,\chi) = (2,5,-7)$. For larger dimension κ, the situation is of course much more complicated, but always $\chi = (-1)^\kappa \mathrm{vol}(\gamma)$.

Compactifications. In algebraic geometry, one normally wants to compactify a given open variety such as $X_{\gamma,t}$ and there are typically many natural ways of doing it.

We already saw the compactification $X^{\mathrm{bcm}}_{\gamma,t}$. It is a hypersurface of degree $\mathrm{vol}(\gamma)$ in the projective space $\mathbb{P}^d$ defined by the first equation of (3.1). On the other hand, for any choice of matrix m with all entries nonnegative, homogenization of (3.2) gives a alternative compactification $\overline{X}_{\gamma,t} \subset \mathbb{P}^d$.

In our continuing example $\gamma = [-5,-2,3,4]$, the plane curve $X^{\mathrm{bcm}}_{\gamma,t}$ has degree seven. In contrast, the plane curve $\overline{X}_{\gamma,t}$ has degree just four, this number arising as the maximum column sum of the matrix m in Table 3.1. Smooth curves in these degrees have genera 15 and 3 respectively. For $t \neq 1$, $X_{\gamma,t}$ has genus 2 so $X^{\mathrm{bcm}}_{\gamma,t}$ must have bad singularities while $\overline{X}_{\gamma,t}$ has just a single node.

Another compactification $X^{\mathrm{BCM}}_{\gamma,t}$ of $X_{\gamma,t}$ is a major focus of [BCM15]. It is typically not smooth, but only has quotient singularities. These singularities are mild in the sense that $X^{\mathrm{BCM}}_{\gamma,t}$ looks smooth from the viewpoint of rational cohomology, and they may be ignored when discussing motives as in the next section.

4 HGMs from cohomology

The HGMs $H(\gamma,t)$ of this survey live in the category $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ of pure motives over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$ defined by André in [And04]. Here we sketch what objects of $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ are in general and define hypergeometric motives $H(\gamma,t)$ in particular. At our level of presentation, the technical changes made by André to render Grothendieck's original vision unconditional are not seen.

Motives on an intuitive level. Let X be a κ-dimensional smooth projective variety over $\mathbb{Q}$, so that $X(\mathbb{C})$ is a compact manifold of real dimension 2κ and all the $H^k(X,\mathbb{Q}) := H^k(X(\mathbb{C}),\mathbb{Q})$ are finite-dimensional rational vector spaces. These vector spaces support many very rich extra structures as we will see in subsequent sections. The formalism of motives lets one treat these extra structures in a parallel way, without going into the details of the extra structures.

"Motive" in this paper officially means "object in the category $\mathcal{M}(\mathbb{Q},\mathbb{Q})$." Full cohomology spaces $H^k(X,\mathbb{Q})$ are motives. For an integer j, their Tate twists $H^k(X,\mathbb{Q}(j))$, as explained later in this section, are motives too. A property of the category $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ is that every motive is the direct sum of irreducible motives. Another property is that any irreducible motive is a direct summand of some $H^k(X,\mathbb{Q}(j))$. The weight of an irreducible motive is important, with summands of $H^k(X,\mathbb{Q}(j))$ having weight $w = k-2j$.

As an example, suppose that $X \subset \mathbb{P}^{\kappa+1}$ is a hypersurface with $\kappa = 2i$ even. Then there is a decomposition

$$H^\kappa(X,\mathbb{Q}) = PH^\kappa(X,\mathbb{Q}) \oplus \mathbb{Q}(-i). \tag{4.1}$$

Here $\mathbb{Q}(-i)$ is a one-dimensional motive; it is spanned by the class in cohomology of $X \cap H$, where H is a linear subvariety of $\mathbb{P}^{\kappa+1}$ of codimension i. The complementary motive $PH^\kappa(X,\mathbb{Q})$ is the classical primitive middle cohomology of X. For generic X, the motive $PH^\kappa(X,\mathbb{Q})$ is irreducible.

Hodge vectors as a key invariant of motives. A leading example of an extra structure comes from Hodge theory, where one represents κ-dimensional cohomology classes via harmonic forms with "p dz_j's and q $d\bar{z}_j$'s in local coordinates." This gives a decomposition of complex vector spaces $H^w(X,\mathbb{C}) = \bigoplus_{p=0}^{w} H^{p,w-p}$, with complex conjugation on coefficients switching $H^{p,q}$ and $H^{q,p}$. The Hodge numbers $h^{p,q} := \dim(H^{p,q})$ therefore satisfy Hodge symmetry $h^{p,q} = h^{q,p}$ and sum to the Betti number $\dim(H^w(X,\mathbb{C}))$.

We systematically present Hodge numbers in vectors $h = (h^{w,0}, h^{w-1,1}, \ldots, h^{1,w-1}, h^{0,w})$. As an example drawn from Section 6, for cubic fourfolds the decomposition (4.1) gives rise to

$$(0,1,21,1,0) = (0,1,20,1,0) + (0,0,1,0,0). \tag{4.2}$$

The Hodge vector notation is very concise: it removes the need for many instances of "$h^{p,q} =$" and the weight $w = p+q$ can be read off as the number of commas.

Tate twisting and Hodge normalization. Passing from $H^k(X,\mathbb{Q}) = H^k(X,\mathbb{Q}(0))$ to general $H^k(X,\mathbb{Q}(j))$ is called "Tate twisting". It is an easy but important part of the motivic formalism. At the level of Hodge numbers, twisting by $j \in \mathbb{Z}$ subtracts j from the Hodge indices p and q. So, in the setting of (4.2), the Hodge numbers of $PH^4(X,\mathbb{Q}(1))$ form the Hodge vector $(h^{2,0}, h^{1,1}, h^{0,2}) = (1, 20, 1)$. Similarly, the second addend of (4.2) comes from $\mathbb{Q}(-2)$ with its unique nonvanishing Hodge number $h^{2,2} = 1$, while a convenient Tate twist is the motive $\mathbb{Q} = H^0(\mathrm{Spec}(\mathbb{Q}),\mathbb{Q})$ with Hodge vector $(h^{0,0}) = (1)$.

Suppose a given nonzero motive $M \subseteq H^k(X,\mathbb{Q})$ has exactly j zeros on each end of its Hodge vector. The corresponding *Hodge-normalized* motive is then $M(j)$ of weight $k - 2j$. So the previous paragraph described the passage from two motives to their Hodge-normalized versions. The purpose of all this Tate twisting, explained in more detail in Section 7, is that a Hodge-normalized motive of weight w is expected to arise directly as a summand in the middle cohomology of a w-dimensional variety.

Definition of HGMs. Let γ be a gamma vector of length $\kappa+3$ and r negative entries, let $t \in \mathbb{Q} - \{0,1\}$ be a specialization point, and let $X_{\gamma,t}$ be the corresponding κ-dimensional canonical variety. Then $H(\gamma,t)$ is the Hodge normalization of the "main piece" $H'(\gamma,t)$ of the compactly supported middle cohomology $H^\kappa_c(X_{\gamma,t},\mathbb{Q})$ of the smooth affine variety $X_{\gamma,t}$.

In more detail, the subquotient $H'(\gamma,t)$ of $H^\kappa_c(X_{\gamma,t},\mathbb{Q})$ is cut out in two steps. First, the contribution of the ambient $(\kappa+1)$-dimensional torus is eliminated to obtain the primitive subspace $PH^\kappa_c(X_{\gamma,t},\mathbb{Q})$. Second, consider any smooth compactification $\overline{X}$ of $X_{\gamma,t}$, or one with at worst mild singularities as mentioned above. Then $H'(\gamma,t)$ is the image of $PH^\kappa_c(X_{\gamma,t},\mathbb{Q})$ under the natural map to $H^\kappa(\overline{X},\mathbb{Q})$. As a quotient of $PH^\kappa_c(X_{\gamma,t},\mathbb{Q})$, the space $H'(\gamma,t)$ is independent of the choice of compactification.

The compactification $X^{BCM}_{\gamma,t}$ is always a permissible choice for $\overline{X}$. Its middle cohomology decomposes as

$$H^\kappa(X^{BCM}_{\gamma,t},\mathbb{Q}) = H'(\gamma,t) \oplus T. \tag{4.3}$$

This decomposition is described at the level of point counts in [BCM15, Thm 1.5]; in particular, the discarded motive T is zero if κ is odd and the sum of $\binom{\kappa+1}{r-1}$ copies of $\mathbb{Q}(-\kappa/2)$ if κ is even. The explicit compactifications that we will use in the sequel are all in Sections 6–8; in these examples, $\overline{X}$ is always just a smooth hypersurface in projective space.

More conceptually, $PH^\kappa_c(X_{\gamma,t},\mathbb{Q})$ is a "mixed motive" of rank $\mathrm{vol}(\gamma)-1$ and the pure motive $H'(\gamma,t)$ in $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ is its weight κ quotient. The passage from $PH^\kappa_c(X_{\gamma,t},\mathbb{Q})$ to $H'(\gamma,t)$ is closely related to the reduction of fractions as in (2.3). In particular, $H'(\gamma,t)$ has rank $n = \deg(Q)$. The full mixed motive $PH^\kappa_c(X_{\gamma,t},\mathbb{Q})$ is itself of great interest, with its lower weight parts playing an important role in deeper studies of hypergeometric motives.

Motivic Galois groups. The formal structure of the category $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ is made clearer by a group-theoretic description. This viewpoint is helpful for the rest of this survey in a general way, but it is taken explicitly only at a few brief points in the sequel. The description centers on a reductive proalgebraic group $\mathbb{G}_\mathbb{Q}$ called the absolute motivic Galois group of $\mathbb{Q}$. This group sits in a short exact sequence with quotient the profinite group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and kernel conjecturally connected. The category $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ is then nothing but the category of representations of $\mathbb{G}_\mathbb{Q}$ on finite-dimensional $\mathbb{Q}$-vector spaces. One can think of $\mathbb{G}_\mathbb{Q}$ as an abstract structure that coordinates more concrete structures on motives. For example $\mathbb{G}_\mathbb{Q}(\mathbb{R})$ contains a copy of $\mathbb{C}^\times$ such that $z \in \mathbb{C}^\times$ acts on any $H^{p,q}$ by $z^p \bar{z}^q$. The image of $\mathbb{G}_\mathbb{Q}$ on a motive M is called the motivic Galois group G_M of M.

Consider now a Hodge-normalized motive M of weight w. For $w = 0$, the representation is conjectured to factor through the quotient $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, so that G_M is finite. For $w \geq 1$, one has an inclusion into a conformal symplectic or orthogonal group according to parity:

$$G_M \subseteq \begin{cases} CSp_n & \text{if } w \text{ is odd}, \\ CO_n & \text{if } w \text{ is even}. \end{cases} \tag{4.4}$$

Here the word "conformal" indicates that scalars have been adjoined to standard symplectic or orthogonal groups to increase their dimension by one. Scalars arise because the bilinear pairing has the form $M \otimes M \to \mathbb{Q}(-w)$, and the action of $\mathbb{G}_\mathbb{Q}$ on $\mathbb{Q}(-w)$ is not trivial. In the case of hypergeometric motives, the pairing is given by a classical construction due to Bezout [JRV14, §3.5].

For generic $t \in \mathbb{Q} - \{0,1\}$, the abstract-looking G_M is determined by the explicit matrix group $\Gamma = \langle h_\infty, h_0 \rangle$ of Section 2. For $w = 0$, the relation is very simply $G_M = \Gamma$. For $w \geq 1$, the group G_M is the Zariski closure of scalars and Γ. Under our standing assumption that $\gcd(\gamma_1,\ldots,\gamma_l) = 1$, a consequence is that equality holds in (4.4). So while all sorts of groups can arise as G_M for general motives M, there is a simple expectation for all hypergeometric motives.

For special t, the motivic Galois group can only become smaller. In the introductory example (1.7), G_M drops from the generic $CSp_2 = GL_2$ only at the three points t where the elliptic curve E_t has potential complex multiplication, namely $t \in \{-1, 1/2, 2\}$. For these t, the motivic Galois group has dimension two rather than four, as it is a normalizer of a maximal torus in GL_2.

5 Hodge numbers

One of the very first things one wants to know about a motive is its Hodge numbers. Fortunately, this desire is easily satisfied for HGMs by an appealing procedure.

Zigzag procedure. The procedure we are about to describe is equivalent to a formula conjectured by Corti and

Golyshev [CG11] and proved by different methods in Fedorov [Fed18] and [RV19]. The procedure is completely combinatorial and only depends on the interlacing pattern of the roots of q_∞ and q_0 in the unit circle.

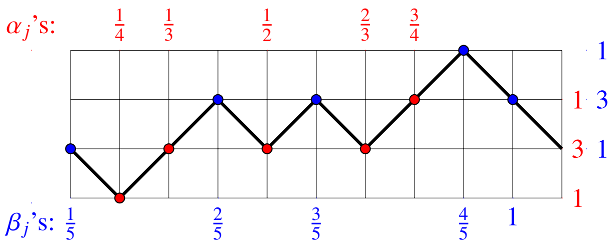

Figure 5.1: The zigzag procedure with input the family parameter $Q = \Phi_2\Phi_3\Phi_4/\Phi_1\Phi_5$ coming from the gamma vector $\gamma = [-4,-3,1,1,5]$ and output the Hodge vector $h = (1,3,1)$

To pass from a family parameter $Q = q_\infty/q_0$ to its Hodge vector h one proceeds as illustrated by Figure 5.1. One orders the parameters α_j and β_j, viewed as elements in say $(0,1]$; for more immediate readability, we associate the colors red and blue to ∞ and 0 respectively. One draws a point at $(0,0)$ corresponding to the smallest parameter in a Cartesian plane. One then proceeds in uniform steps from left to right, drawing a point for each parameter and then moving diagonally upwards after red points and diagonally downwards after blue points. One focuses on one color or the other, counting the number of points on horizontal lines. The numbers obtained form the Hodge vector h. The red and blue dots yield the same Hodge vector but contain more information. They may be used to describe the limiting mixed Hodge structure at $t = \infty$ and 0 respectively.

The completely intertwined case. Complete intertwining of the α_j and β_j gives the extreme where the resulting Hodge vector is just (n). Beukers and Heckman [BH89] proved that complete intertwining is exactly the condition one needs for the monodromy group $\langle h_\infty, h_0\rangle$ to be finite. They also established the complete list of such pairs (α,β). Actually they, like Schwarz who famously treated the $n = 2$ case more than a century earlier, worked without our standing assumption $E = \mathbb{Q}$. Then one needs to require complete intertwining of all the natural conjugates of (α,β) and the list obtained is longer.

In our setting of $E = \mathbb{Q}$, the corresponding gamma vectors are of odd lengths 3 to 9. There are infinite collections of length 3 and 5 given by coprime positive integers a, b:

$$\begin{aligned} (i) \quad & [-(a+b), a, b], \\ (ii) \quad & [-2(a+b), -a, 2a, b, a+b], \\ (iii) \quad & [-2a, -2b, a, b, a+b]. \end{aligned} \tag{5.1}$$

Here and always, we allow ourselves to write the components of gamma vectors in a nonstandard order when convenient, the weakly increasing ordering just being a normalization. Similarly, whenever we discuss classifications, we omit consideration of $-\gamma$ if γ has already been listed. In case (i), the canonical variety consists of just $a+b$ points. Removing a variable, the BCM presentation takes the form

$$X_{a,b,t} : y^a(1-y)^b - \frac{a^a b^b}{(a+b)^{a+b}} t = 0.$$

For $b = 1$ this presentation is already trinomial; in general, one has to make a non-trivial change of variables to pass to the trinomial presentation of $X_{a,b,t}$ given by a toric model.

Beyond the closely related collections (i) - (iii), there are only finitely many further γ, all related to Weyl groups. [BH89, Table 8.3] says that, modulo the quadratic twisting operation $Q(T) \mapsto Q(-T)$, there are just one, five, five, and fifteen respectively for the groups $W(F_4)$, $W(E_6)$, $W(E_7)$, and $W(E_8)$. A $W(E_6)$ case is discussed in Section 7 and the remaining $W(E_n)$ cases are similarly treated in [Rob18].

The completely separated case. Complete separation of the α_j and β_j gives the extreme where the resulting Hodge vector is $(1,1,\ldots,1,1)$. The subcase where $q_0 = (T-1)^n$ has the simplifying feature that h_0 consists of a single Jordan block. Families in this subcase have received particular attention in the physics literature; the condition is sometimes verbalized as MUM, for *maximal unipotent monodromy*.

Classification of families in the completely separated case is easier than in the completely intertwined case. It becomes trivial in the MUM subcase because q_∞ is arbitrary except for the fact that it contains no factors of $(T-1)$. Accordingly, the number c_n of rank n families in the MUM subcase is given by a generating function

$$\begin{aligned} \sum_{n=0}^{\infty} c_n x^n &= \prod_{k=2}^{\infty} \frac{1}{1-x^{\phi(k)}} \\ &= 1 + x + 4x^2 + 4x^3 + 14x^4 + 14x^5 + \cdots \end{aligned} \tag{5.2}$$

Always $c_{2j} = c_{2j+1}$ as under the MUM restriction multiplying by $(T+1)/(T-1)$ gives a bijection on family parameters. Information about the list underlying $c_4 = 14$ is in [RV03].

Signature and the Magma implementation. A motive has a signature σ, which is the trace of complex conjugation. For odd weight motives, it is always zero. For even weight HGMs $H(Q,t)$, it depends only on Q and the interval $(-\infty,0)$, $(0,1)$, or $(1,\infty)$ in which t lies. *Magma*'s command `HodgeStructure` returns both the Hodge vector and the signature in coded form. To see just the Hodge vector clearly, one can implement `Q` as in (2.4) and type

```
HodgeVector(HodgeStructure(Q,2));
```

For example, from the gamma vector $[-21,1,2,3,4,5,6]$ one gets the Hodge vector $(1,2,12,2,1)$.

6 Projective Hypersurfaces

Here we realize some HGMs in the cohomology of the most classical varieties of all, smooth hypersurfaces in projective space.

Hodge numbers. Let $X \subset \mathbb{P}^{\kappa+1}$ be a smooth hypersurface of degree δ. Let $PH^\kappa(X,\mathbb{Q})$ be the primitive part of its middle cohomology. So this primitive part is the entire middle cohomology if κ is odd and a subspace of codimension one as in (4.1) if κ is even.

Hirzebruch gave a formula for the Hodge numbers of $PH^\kappa(X,\mathbb{Q})$ as a function of κ and δ. For example, the sum of the Hodge numbers and first Hodge number are respectively

$$b_\kappa = \frac{(\delta-1)^{\kappa+2} + (-1)^\kappa(\delta-1)}{\delta}, \qquad h^{\kappa,0} = \binom{\delta-1}{\kappa+1}. \tag{6.1}$$

These special cases and Hodge symmetry are sufficient to give Hodge vectors when $\kappa \leq 3$:

δ	Curves in $\mathbb{P}^2$	Surfaces in $\mathbb{P}^3$	Threefolds in $\mathbb{P}^4$
3	$(1,1)$	$(0,6,0)$	$(0,5,5,0)$
4	$(3,3)$	$(1,19,1)$	$(0,30,30,0)$
5	$(6,6)$	$(4,44,4)$	$(1,101,101,1)$
6	$(10,10)$	$(10,85,10)$	$(5,255,255,5)$.

(6.2)

For $\kappa = 1$, either part of (6.1) reduces to the genus formula for smooth plane curves, $g = (\delta-1)(\delta-2)/2$.

One example for every (δ,κ)**.** Let $\delta = e+1 \geq 3$ be a desired degree and let κ be a desired dimension. Define

$$\gamma = [1, -e, e^2, \ldots, (-e)^\kappa, (-e)^{\kappa+1} - 1, \frac{(-e)^{\kappa+2} + e}{e+1}]. \tag{6.3}$$

The toric procedure illustrated by Table 3.1 yields the completed canonical variety:

$$X_t: \quad u x_{\kappa+1} x_1^e + \sum_{i=2}^{\kappa+2} x_{i-1} x_i^e + x_{\kappa+2}^{e+1} = 0. \tag{6.4}$$

The necessary orthogonality relations on each variable's exponents are illustrated by the case of cubic fourfolds where $\gamma = [1,-2,4,-8,16,-33,22]$. Then (6.4) becomes

$$X_t : u x_5 x_1^2 + x_1 x_2^2 + x_2 x_3^2 + x_3 x_4^2 + x_4 x_5^2 + x_5 x_6^2 + x_6^3 = 0.$$

For x_5, the relation is that $m_{5*} = (1,0,0,0,2,1,0)$ is orthogonal to γ. In general, partial derivatives of (6.4) are very simple since the row vectors m_{i*} have just two nonzero entries, except for the case $i = \kappa+1$ and its three nonzero entries. It is then a pleasant exercise to check via the Jacobian criterion that X_t is smooth for $t \in \mathbb{C}^\times - \{1\}$.

The degree of the rational function Q determined by γ can be computed uniformly in (δ,κ) as the cancellations to be analyzed are very structured. This degree agrees with the Betti number b_κ from (6.1). Thus $H(\gamma,t)$ is the full primitive middle cohomology of X_t, while *a priori* it might have been a proper subspace. The zigzag procedure for computing Hodge numbers must agree in the end with the Hirzebruch formula. The reader might want to check the above case of cubic fourfolds, where Hirzebruch's full formula gives $(0,1,20,1,0)$.

All examples for a given (δ,κ)**.** An interesting problem is to find all γ which give projective smooth κ-folds of degree δ. For small parameters, this problem can be solved by direct computation. For example, consider $(\delta,\kappa) = (3,4)$, thus cubic fourfolds. In this case, one has the standardization $[-33,-8,-2,1,4,16,22]$ of the above example, and then exactly ten more:

$$\begin{array}{ll} [-48,-15,-12,5,16,24,30], & [-36,-9,-4,3,8,18,20], \\ [-48,-12,-3,1,6,24,32], & [-33,-16,-4,2,8,11,32], \\ [-48,-12,-3,6,16,17,24], & [-33,-10,-7,5,11,14,20], \\ [-36,-16,-9,3,8,18,32], & [-33,-4,-1,2,8,11,17], \\ [-36,-9,-8,4,15,16,18], & [-21,-20,-16,7,8,10,32]. \end{array}$$

7 Dimension reduction

An HGM $H(\gamma,t)$ is defined in terms of a κ-dimensional variety but its Hodge vector $(h^{w,0},\ldots,h^{0,w})$ raises the question of whether it also comes from a variety of dimension $w = \kappa - 2j$. The exterior zeros for low degree projective hypersurfaces as illustrated in (6.2) raise the same question. The generalized Hodge conjecture says that this dimension reduction is always possible. We illustrate here some of the appealing geometry that arises from reducing dimension.

Reduction to points. When $w = 0$ the reduction to dimension zero is possible in all cases. For example, $\gamma = [-12,-3,1,6,8]$ corresponds to entry 45 on the Beukers-Heckman list [BH89, Table 8.3]. Formula (3.2) then gives a family X_t of cubic surfaces. An equation whose roots correspond to the famous twenty-seven lines on X_t is

$$2^4 t x^3 (x^2-3)^{12} - 3^9 (x^3 - 3x^2 + x + 1)^8 (x-2) = 0. \tag{7.1}$$

The Galois group of this polynomial $g(t,x)$ for generic $t \in \mathbb{Q}^\times - \{1\}$ is $W(E_6)$. It has $51840 = 2^7 3^4 5$ elements and is also the monodromy group $\Gamma = \langle h_\infty, h_0 \rangle$.

Reduction via splicing. Suppose γ can be written as the concatenation of two lists each summing to zero. Then one can use a general splicing technique from [BCM15, §6] to reduce the dimension by two. This technique is behind the scenes even of our introduction: in the family of examples there, the canonical varieties for $[-2,-2,1,1,1,1]$ are three-dimensional, although the more familiar source varieties are just the Legendre curves (1.3).

For an example complicated enough to be representative of the general case, take $\gamma = [-12,-3,-2,1,1,1,6,8]$ so that the canonical variety has dimension $\kappa = 5$. The Hodge vector is just $(3,3)$, so one would like to realize $H(\gamma,t)$ in the cohomology of a curve.

Splicing is possible because both $[-12, -3, 1, 6, 8]$ and $[-2, 1, 1]$ sum to zero. No further splicing is possible, but fortunately we have just treated the first sublist by other means. Splicing corresponds to taking a fiber product over the t-line which in turn corresponds to just multiplying rational functions. In our case, solving (7.1) for t to get the first factor, the dimension-reduced variety is given by

$$\frac{3^9(x-2)\left(x^3-3x^2+x+1\right)^8}{2^4x^3\left(x^2-3\right)^{12}} \cdot \frac{2y-1}{y^2} = t. \tag{7.2}$$

The variable y from $[-2, 1, 1]$ enters only quadratically and so (7.2) defines a double cover of the x-line. Taking the discriminant with respect to y and removing unneeded square factors presents this hyperelliptic curve in standard form:

$$z^2 = -3(x-2)g(t,x). \tag{7.3}$$

Here $g(t,x)$ is the degree 27 polynomial from (7.1). As the right side of (7.3) has degree 28, this curve has genus 13.

Comparison of source varieties. The middle cohomology of the dimension-reduced varieties (7.1) and (7.3), contains not only the desired motives, with Hodge vectors (6) and $(3,3)$ respectively, but also "parasitical" motives, with Hodge vectors (21) and $(10,10)$. In this regard, they are less attractive than the original canonical varieties. HGMs provide many illustrations like these two of the motivic principle that a motive M comes from many varieties, and often no particular variety should be viewed as the best source.

8 Distribution of Hodge vectors

In this section, we explain one of the great features of HGMs: they represent many Hodge vectors.

Completeness in ranks ≤ 19. By applying the zigzag algorithm to all possible family parameters Q in low degrees, we have verified the following fact. *Let $h = (h^{w,0}, \cdots, h^{0,w})$ be a vector of positive integers satisfying $h^{q,p} = h^{p,q}$ for all $p+q = w$ and let $n = \sum_{i=0}^{w} h^{p,w-p}$. Then if $n \leq 19$ there exists at least one family of HGMs with Hodge vector h.*

Many families per Hodge vector in ranks ≤ 100. In ranks 20 to 23, the only vectors not realized by a family of HGMs are

$$\begin{array}{rc} 20: & (6,1,1,1,2,1,1,1,6), \\ 22: & (6,1,1,1,1,2,1,1,1,1,6), \\ 22: & (4,1,2,1,1,1,2,1,1,1,2,1,4), \\ 23: & (1,21,1). \end{array}$$

Table 8.1 gives a fuller sense of the situation for $n = 24$, where there are about 460 million family parameters. It gives the extremes of the list of 4096 possible Hodge vectors h, sorted by how many families realize h.

h	#
(9, 1, 1, 2, 1, 1, 9)	0
(7, 1, 1, 1, 1, 2, 1, 1, 1, 1, 7)	0
(1, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 6, 1)	0
(4, 1, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 4)	0
(5, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 5)	0
(6, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 6)	0
(4, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 4)	0
(4, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 4)	0
(6, 2, 1, 1, 1, 2, 1, 1, 1, 2, 6)	2
(8, 1, 1, 1, 2, 1, 1, 1, 8)	4
(1, 22, 1)	4
(8, 1, 1, 4, 1, 1, 8)	6
⋮	⋮
(1, 2, 4, 5, 5, 4, 2, 1)	7982874
(2, 4, 6, 6, 4, 2)	9504072
(1, 4, 7, 7, 4, 1)	9905208

Table 8.1: Hodge vectors with total 24 and their number of hypergeometric realizations

The ratio of the numbers reported in the previous paragraph say that the number of family parameters per Hodge vector in degree 24 is about 113,000. This ratio increases to a maximum at $n = 58$ where it is about four million. It then decreases to zero, with some approximate sample values being two million for $n = 100$ but only 0.00001 for $n = 300$. These numbers are computed via generating functions, similar to (5.2) but more complicated.

Perspective. Comparison with Section 6 offers some perspective on the general inverse problem of finding an irreducible motive $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ with a given Hodge vector. From (6.1)–(6.2), one sees that the set of Hodge vectors coming from hypersurfaces is very sparse. When one looks at broader standard classes of varieties, such as complete intersections in projective spaces, more Hodge vectors arise, but they all have the same rough form: bunched in the middle. *Ad hoc* techniques, such as reducing Hodge numbers by imposing singularities, give many more Hodge vectors. But for many h, it does not seem easy to find a corresponding motive in this geometric way. For example, imposing k ordinary double points on a sextic surface reduces the Hodge vector to $(10, 85-k, 10)$. However the family of sextic surfaces is only 68-dimensional, and so it would it seem to be difficult to get down to e.g. $(10,1,10)$. There does not seem to be even a conjectural expectation of which Hodge vectors arise from irreducible motives in $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$.

The cases $(1, b, 1)$. One could go into much more detail about the families behind any given Hodge vector. Here we say a little more about the cases $(1,b,1)$. The γ giving Hodge vectors of the form $(1,b,1)$ typically have canonical dimension $\kappa = \dim(X_{\gamma,t})$ greater than two, posing instances of the dimension reduction problem. If $b \leq 19$, then the moduli theory of $K3$ surfaces says that there is at least one family Y_t of $K3$ surfaces also realizing $H(\gamma, t)$. Finding such a family is a challenge.

Cases with $b \geq 20$ present a greater challenge, as they cannot be realized by K3 surfaces. There are seventy-two parameters giving $(1, 20, 1)$. None of the eleven listed at the end of Section 6 can be spliced, underscoring the difficulty of dimension reduction. One of the four gamma vectors giving $(1, 22, 1)$ has canonical dimension eight, namely $[-60, -5, -4, -3, -2, 8, 9, 10, 12, 15, 20]$. The other three have canonical dimension ten:

$$[-66, -11, -6, -5, -4, -4, 1, 2, 8, 12, 18, 22, 33],$$
$$[-60, -15, -9, -6, -4, -2, 3, 5, 8, 12, 18, 20, 30],$$
$$[-33, -10, -6, -4, -4, -1, 2, 2, 5, \ 8, 11, 12, 18].$$

In all four cases, there are many ways to splice, but no path to a surface.

9 Special and semi HGMs

We have so far been excluding the singular specialization point $t = 1$ from consideration. Now we explain how it yields a particularly interesting motive $H(Q, 1) \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$. We also explain how other interesting motives arise when the family parameter Q is reflexive, in the sense of satisfying $Q(-T) = Q(T)^{-1}$.

Interior zeros. A Hodge-normalized motive $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ of weight w has Hodge vector $h = (h^{w,0}, \ldots, h^{0,w})$ with $h^{w,0} = h^{0,w} > 0$. But for the Hodge vectors explicitly considered so far, the remaining numbers $h^{p,w-p}$ are also positive. There is a reason for this restriction: Griffiths transversality says that any collection of motives moving in a family with irreducible monodromy group has Hodge vector with no interior zeros. Special and semi HGMs do not move in families, and they include cases with interior zeros.

Special HGMs. The way to account for the double point on the canonical variety $X_{\gamma,1}$ is to first of all take inertial invariants with respect to the monodromy operator h_1. In the orthogonal case where w is even, h_1 is conjugate to $\mathrm{diag}(-1, 1, \ldots, 1)$. Here taking inertial invariants already gives the right motive $H(Q, 1)$. Its Hodge vector differs from the Hodge vector of the other $H(Q, t)$ only in that $h^{w/2,w/2}$ is decreased by 1. In the symplectic case where w is odd, h_1 is conjugate to $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \oplus \mathrm{diag}(1, \ldots, 1)$. Here the motive of inertial invariants leaves our context because it is mixed; quotienting out by its submotive of weight $w-1$ and rank 1 gives $H(Q, 1)$. Its Hodge vector now comes from the generic one by decreasing the two central Hodge numbers by 1. These drops obviously can cause interior zeros, as in $(10, 1, 10) \to (10, 0, 10)$ or $(1, 1, 1, 1, 1, 1) \to (1, 1, 0, 0, 1, 1)$.

Semi HGMs. The interest in reflexive parameters is that nongeneric behavior is forced at the fixed points ± 1 of $t \mapsto 1/t$. The motive $H(Q, (-1)^n)$ is a direct sum of two motives in $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$ of roughly equal rank. We call the summands semi HGMs and their Hodge vectors can have many interior zeros. For example, the summands of $H(\Phi_2^{16}/\Phi_1^{16}, 1)$ are studied in [Rob19] and the two Hodge vectors are

$$\begin{gathered}(1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1),\\ (1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1).\end{gathered} \tag{9.1}$$

For $H(Q, -(-1)^n)$, there is no forced decomposition, but the motivic Galois group becomes smaller.

10 Point counts

We now turn to arithmetic. The point counts of this section form the principal raw material from which the L-functions studied in the remaining sections are built.

Background. Let X be a smooth projective variety over $\mathbb{Q}$. Then for all primes p outside a finite set S, the equations defining X have good reduction and so define a smooth projective variety over $\mathbb{F}_p$. For any power $q = p^e$, one has the finite set of solutions $X(\mathbb{F}_q)$ to the defining equations. The key invariants that need to be input into the motivic formalism are the cardinalities $|X(\mathbb{F}_q)|$, and famous results of Grothendieck, Deligne, and others provide the tools.

The vector spaces $H^k(X, \mathbb{Q})$ do not see that X is defined over $\mathbb{Q}$. The arithmetic origin of X yields extra structure as follows. For any prime ℓ, one can extend coefficients to obtain vector spaces $H^k(X, \mathbb{Q}_\ell)$ over the field $\mathbb{Q}_\ell$ of ℓ-adic numbers. In sharp contrast to $H^k(X, \mathbb{Q})$ itself, the spaces $H^k(X, \mathbb{Q}_\ell)$ can also be defined purely algebraically, via the theory of étale cohomology. Via this connection, the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H^k(X, \mathbb{Q}_\ell)$.

For every prime p, the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contains a Frobenius element Fr_p, well-defined up to conjugation and modulo inertia subgroups; the invariants we are describing are independent of the choice of Fr_p. For any power $q = p^e$ of a prime $p \notin S$, and any $\ell \neq p$, one has the trace of the operator $\mathrm{Fr}_q = \mathrm{Fr}_p^e$ acting on $H^k(X, \mathbb{Q}_\ell)$. These ℓ-adic numbers are in fact rational and independent of ℓ. We emphasize the independence of ℓ by denoting them $\mathrm{Tr}(\mathrm{Fr}_q | H^k(X, \mathbb{Q}))$. The connection with point counts is the Lefschetz trace formula: $|X(\mathbb{F}_q)| = \sum_k (-1)^k \, \mathrm{Tr}(\mathrm{Fr}_q | H^k(X, \mathbb{Q}))$. The left side for fixed p and varying e determines the summands on the right side because the complex eigenvalues of Fr_p on $H^k(X, \mathbb{Q})$ have absolute value $p^{k/2}$.

Much of this transfers formally to the motivic setting. Thus for a motive M and a prime ℓ, there is an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the corresponding ℓ-adic vector space M_ℓ. As an example, for the motives $\mathbb{Q}(-j)$ the action is through the abelianization of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and Frobenius traces are very simply $\mathrm{Tr}(\mathrm{Fr}_p | \mathbb{Q}(-j)) = p^j$. In general, this action has image in $G_M(\mathbb{Q}_\ell)$. In fact, the Tate conjecture predicts that the $\mathbb{Q}_\ell$-Zariski closure of the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is all of $G_M(\mathbb{Q}_\ell)$.

One technical problem with André's category $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ is that the projectors used to define motives are not known to come from algebraic cycles. As a consequence, for a general $M \in \mathcal{M}(\mathbb{Q},\mathbb{Q})$ the above compatibility of Frobenius traces is not known. However this problem does not arise for hypergeometric motives, because they are essentially the entire middle cohomology of varieties, by (4.3). Accordingly, one has well-defined rational numbers $\mathrm{Tr}(\mathrm{Fr}_q|H(\gamma,t))$. There are similar technical problems at the primes $p \in S$, but they do not affect our computations and we will ignore them.

Wild, tame, and good primes. Returning now to very concrete considerations, we sort primes for a parameter (γ,t) as follows. A prime p is *wild* if it divides a γ_j. For $t \neq 1$, a prime p is *tame* if it is not wild but it divides either the numerator of t, the denominator of t, or the numerator of $t-1$; these last three conditions say that t is p-adically close to the special points 0, 1, and ∞ respectively. For $t = 1$, no primes are tame. We say that a prime is *bad* if it is either wild or tame, and all other primes are *good*.

Split powers of a good prime p**.** A power q of a good prime p is called split for γ if $q \equiv 1 \bmod m$, where m is the least common multiple of the γ_j. One then has a collection of Jacobi sums indexed by characters χ of $\mathbb{F}_q^\times$:

$$J(\gamma,\chi) := \prod_{j=1}^{n} g(\omega^{\alpha_j(q-1)}\chi,\psi)\overline{g(\omega^{\beta_j(q-1)}\chi,\psi)}.$$

Here $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ is any nonzero additive character, ω is any generator of the group of multiplicative characters, (α,β) underlies γ as in Section 2, and $g(\rho,\psi) = \sum_{t\in\mathbb{F}_q^\times} \rho(t)\psi(t)$ is the standard Gauss sum. The desired quantity is then given by a sum due to Katz [Kat90, §8.2]. Renormalizing to fit our conventions, it is

$$\mathrm{Tr}(\mathrm{Fr}_q|H(\gamma,t)) = \frac{q^{\phi_0}}{1-q}\sum_{\chi}\frac{J(\gamma,\chi)}{J(\gamma,1)}\chi(t). \tag{10.1}$$

Here ϕ_0 is the vertical coordinate of a lowest point on the zigzag diagram of γ, e.g. $\phi_0 = -1$ in Figure 5.1.

General powers of a good prime p**.** The Gross-Koblitz formula lets one replace the above Gauss sums by values of the p-adic gamma function. This is both a computational improvement and extends (10.1) from split q to all powers of any good prime. With this method, the desired integers $\mathrm{Tr}(\mathrm{Fr}_q|H(\gamma,t))$ are first approximated p-adically. Errors are under control and exact values are determined from sufficiently good approximations. See [BCM15] for a closely related approach to the essential numbers $\mathrm{Tr}(\mathrm{Fr}_q|H(\gamma,t))$ and references to earlier contributions.

11 Frobenius polynomials

Frobenius polynomials are a concise way of packaging the point counts of the preceding section. They play the leading role in the formula for L-functions of the next section. After saying what they are, this section explains several reasons why they are useful, even before one gets to L-functions.

Capturing point counts. Consider the numbers $c_{p,e} = \mathrm{Tr}(\mathrm{Fr}_p^e|M) \in \mathbb{Q}$ for a fixed motive $M \in \mathcal{M}(\mathbb{Q},\mathbb{Q})$ of rank n, a fixed good prime p, and varying e. They can be captured in a single degree n polynomial $F_p(M,x) = \det(1-\mathrm{Fr}_p x|M)$. The relation, which comes from summing the geometric series belonging to each of the n eigenvalues, is

$$\exp\left(\sum_{e=1}^{\infty}\frac{c_{p,e}}{e}x^e\right) = \frac{1}{F_p(M,x)}. \tag{11.1}$$

Write

$$F_p(M,x) = 1 + a_{p,1}x + \cdots + a_{p,n-1}x^{n-1} + a_{p,n}x^n.$$

Then the $c_{p,e}$ for $e \leq k$ determine $a_{p,k}$. Thus the $c_{p,e}$ for $e \leq n$ determine $F_p(M,x)$. But, even better, Poincaré duality on a source variety ultimately implies that one has $a_{p,e} = \epsilon(p)a_{p,n-e}p^{(n-2e)w/2}$ for a sign $\epsilon(p)$. For HGMs, this sign is known and in fact always 1 when w is odd. So $F_p(M,x)$ can be computed using only $c_{p,e}$ for $e \leq \lfloor n/2 \rfloor$.

Weight as a measure of complexity. Indexing by weight w, consider as examples the rank six family parameters

$$Q_0 = \frac{\Phi_3\Phi_{12}}{\Phi_1\Phi_2\Phi_8}, \qquad Q_1 = \frac{\Phi_3\Phi_{12}}{\Phi_1^2\Phi_8}, \qquad Q_5 = \frac{\Phi_3^3}{\Phi_1^6}. \tag{11.2}$$

The first two are the families from Section 7, with Hodge vectors respectively (6) and (3, 3); the last one has Hodge vector (1, 1, 1, 1, 1, 1). Specializing at a randomly chosen common point gives motives $M_{6,w} = H(Q_w, 3/2)$.

After the required initialization of a variable x by `_<x>:=PolynomialRing(Integers())`, and after inputting `Q`w as in (2.4), Magma quickly gives some Frobenius polynomials via e.g. `EulerFactor(Q0,3/2,5)`:

$$\begin{aligned}
F_5(M_{6,0},x) &= 1 - x - x^5 + x^6,\\
F_7(M_{6,0},x) &= 1 \qquad\qquad - x^6,\\[1em]
F_5(M_{6,1},x) &= 1 + x + 6x^2 + 16x^3 + \cdots,\\
F_7(M_{6,1},x) &= 1 - 2x + 12x^2 - 28x^3 + \cdots,\\[1em]
F_5(M_{6,5},x) &= 1 - 9x + 5{\cdot}156x^2 - 5^3{\cdot}2556x^3 + \cdots,\\
F_7(M_{6,5},x) &= 1 + 12x + 7{\cdot}888x^2 + 7^3{\cdot}1816x^3 + \cdots.
\end{aligned}$$

The coefficients in these displays illustrate a basic motivic principle: as weight increases, Hodge-normalized motives of a given rank n become more complicated.

Congruences. Reduced to $\mathbb{F}_\ell$, the numbers $a_{p,k}$ for $p \neq \ell$ depend only on the mod ℓ Galois representation associated to M. In our examples, suppose one kills $\ell = 2$ in (11.2) by replacing all $\Phi_{2^j a}$ by $\Phi_a^{\phi(2^j)}$. Then Q_0 and Q_1 both become Q_5. This agreement implies that $F_p(M_{6,w}, x) \in \mathbb{F}_\ell[x]$ is independent of w. This independence can be seen for the primes 5 and 7 in the displayed Frobenius polynomials. The analogous congruences hold for any ℓ, when one changes our Tate twist convention to make the weight of $H(Q, t)$ the number of integers among the α_i and β_j, minus one. This web of congruences, like the web corresponding to splicing considered in Section 7, makes it clear that HGMs constitute a natural collection of motives.

Finite Galois groups. Frobenius polynomials render Galois-theoretic aspects of the situation very concrete. As a warm-up, consider Q_0 as a representative of the relatively familiar case of ordinary Galois theory. Here the ℓ-adic representations all come from a single representation $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to W(E_6) \subset GL_6(\mathbb{Q})$. Let λ_p be the partition of 27 obtained by taking the degrees of the irreducible factors of $g(3/2, x)$ from (7.1). Then the twenty-five possibilities for the pair (λ_p, F_p) correspond to the twenty-five conjugacy classes in the finite group $W(E_6)$. If one can collect enough classes, then one can conclude that the image G is all of $W(E_6)$. In our example $t = 3/2$, the above primes 5 and 7 give $(5^5 1^2, 1 - x - x^5 + x^6)$ and $(6^4 3, 1 - x^6)$ respectively. In ATLAS notation, these are the classes $5A$ and $6I$. They do not quite suffice to prove $G = W(E_6)$. But the prime 11 gives the class $12C$ and since no maximal subgroup contains elements from $5A$, $6I$, and $12C$, indeed $G = W(E_6)$.

Infinite Galois groups. The cases Q_1 and Q_5 are beyond classical Galois theory as the motivic Galois groups have positive dimension. But the situation remains quite similar. Consider for example odd weight motives of rank $n = 2r$ so that G is in the conformal symplectic group CSp_n. The Weyl group of CSp_n is the hyperoctahedral group $W(C_r)$ of signed permutation matrices, with order $2^r r!$. A separable $F_p(M, x)$, being conformally palindromic, has Galois group within $W(C_r)$. If it has Galois group all of $W(C_r)$ then G necessarily contains a certain twisted maximal torus. Suppose a second prime p' satisfies the same condition and moreover the joint Galois group of $F_p(M, x) F_{p'}(M, x)$ is all of $W(C_r) \times W(C_r)$. Then G contains two maximal tori which are sufficiently different to force $G = CSp_n$, by the classification of subgroups containing a maximal torus.

To analyze a given motive, the necessary computations can be done using Magma's `GaloisGroup` command. The order of the Galois group of $F_p(M_{6,1}, x)$ is 16, 16, 4, 48, 48 for $p = 5, 7, 11, 13, 17$, and the pair $(p, p') = (13, 17)$ satisfies the criterion. For $5 \le p < 100$, all $F_p(M_{6,5}, x)$ have Galois group $W(C_3)$ except $p = 13$. Excluding 13, all $\binom{22}{2} = 231$ pairs (p, p') satisfy the criterion. In general, it becomes easier to establish genericity as the weight increases, a reflection of the growth in complexity discussed above.

Applying this two-prime technique to the special and semi HGMs of Section 9 suggests that almost always their motivic Galois groups are as big as possible. In particular, the exotic Hodge vectors with interior zeros arising there indeed come from irreducible motives. Details in the case (9.1) are given in [Rob19].

12 L-functions

We now finally define L-functions and illustrate how everything works by some numerical computations.

Local invariants. Let $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ be a motive of rank n and weight w, having bad reduction within a finite set S of primes. We have discussed two types of local invariants associated to M. Corresponding to the place ∞ of $\mathbb{Q}$ is the Hodge vector $h = (h^{w,0}, \ldots, h^{0,w})$ with total n, and also a signature σ. Corresponding to a prime $p \notin S$ is the degree n Frobenius polynomial $F_p(M, x)$. For primes $p \in S$, there is also a Frobenius polynomial $F_p(M, x)$, now of degree $\le n$, and moreover a conductor exponent $c_p \ge n - \deg(F_p(M, x))$, both to be discussed shortly. The conductor of M, which can be viewed as quantifying the severity of its bad reduction, is the integer $N = \prod_{p \in S} p^{c_p}$.

Formal products. The local invariants can be combined into a holomorphic function in the right half-plane $\mathrm{Re}(s) > \frac{w}{2} + 1$, called the completed L-function of M:

$$\Lambda(M, s) = N^{s/2} \Gamma_{h,\sigma}(s) \prod_p \frac{1}{F_p(M, p^{-s})}. \tag{12.1}$$

The product over primes alone is the L-function $L(M, s)$, while the remaining factors give the completion. The infinity factor is given by an explicit formula:

$$\Gamma_{h,\sigma}(s) = \Gamma_\mathbb{R}(s - \frac{w}{2})^{h_+} \Gamma_\mathbb{R}(s - \frac{w}{2} + 1)^{h_-} \prod_{p<q} \Gamma_\mathbb{C}(s - p)^{h^{p,q}}. \tag{12.2}$$

Here $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. The factors involving $h_\pm = (h^{w/2,w/2} \pm (-1)^{w/2}\sigma)/2$ only appear when w is even; in the common case that $\sigma = 0$, they can be replaced by $\Gamma_\mathbb{C}(s - \frac{w}{2})^{h^{w/2,w/2}/2}$, by the duplication formula.

Both the L-function and the completing factor are multiplicative in M so that $\Lambda(M_1 \oplus M_2, s) = \Lambda(M_1, s)\Lambda(M_2, s)$. Another simple aspect of the formalism is that Tate twists correspond to shifts: $\Lambda(M(j), s) = \Lambda(M, s + j)$.

Expected analytic properties. The L-function $L(\mathbb{Q}, s)$ of the unital motive $\mathbb{Q}$ is just the Riemann zeta function $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$, and the completing factor is $\Gamma_\mathbb{R}(s)$. Riemann established that $\Lambda(\mathbb{Q}, s)$ has a meromorphic continuation to the whole s-plane, with poles only at 0 and 1; moreover he proved that $\Lambda(\mathbb{Q}, 1 - s) = \Lambda(\mathbb{Q}, s)$. The product $\Lambda(M, s)$ is expected to have similar analytic properties. First, for M irreducible and not of the form $\mathbb{Q}(j)$, there should be an analytic continuation to the entire s-plane, bounded in vertical strips. Second, always

$$\Lambda(M, w + 1 - s) = \epsilon \Lambda(M, s), \tag{12.3}$$

for some sign ϵ.

Determining invariants at bad primes. One approach to the conductor exponents c_p and Frobenius polynomials $F_p(M,x)$ associated to bad primes p is to compute them directly by studying the bad reduction of an underlying variety. For an HGM $H(Q,t)$, Magma takes this approach for primes which are tame for (Q,t), as sketched in Section 13.

A very different approach uses the fact that the list of possible $(c_p, F_p(M,x))$ for a given prime p is finite, and the product (12.1) has the conjectured analytic properties for at most one member of the product list. The current state of HGMs for the wild primes of Q mixes the two approaches: we first greatly reduce the length of the lists by using proved and conjectured general facts. Then we search within the much smaller product list for the right quantities.

Our view is that numerical computations such as those that follow in this section and Section 15 admit only one plausible interpretation: the bad factors have been properly identified and the analytic properties indeed hold. However rigorous confirmation does not seem to be in sight at the moment, despite the progress described in Section 14.

A rank four example. For Q of degree ≤ 6 and $t = 1$, Watkins numerically identified all the bad quantities, so that the corresponding L-functions are immediately accessible on Magma. For example, take the family parameter to be $Q = \Phi_2^2\Phi_{12}/\Phi_{18}$, implemented as always by modifying (2.4). At the specialization point $t = 1$, the Hodge vector is $(1,1,1,1)$. The corresponding L-function, set up so that calculations are done with 10 digits of precision, is

```
L := LSeries(Q,1:Precision:=10);
```

The information at bad primes stored in Magma is revealed by `EulerFactor(L,p)` and `Conductor(L)` to be $F_2(M,x) = 1 + 2x$, $F_3(M,x) = 1$, and $N = 2^6 3^9$. The sign ϵ is calculated numerically, with `Sign(L)` returning -1.000000000. So the order of vanishing of $L(M,s)$ at the central point $s = 2$ should be odd. This order is apparently three since

```
Evaluate(L,2:Derivative:=1);
```

returns zero to ten decimal places, but the same command with 1 replaced by 3 returns 51.72756346.

A rank six example. More typically, Magma does not know $F_p(M,x)$ and c_p for wild primes p and one needs to input this information. As an example, take $M = H(\Phi_3^4/\Phi_1^8, 1)$ with Hodge vector $(1,1,1,0,0,1,1,1)$. The only prime bad for the data is $p = 3$. A good first guess is that $F_3(M,x)$ is just the constant 1. A short search over some possible c_3 is implemented after redefining `Q` by

```
[CFENew(LSeries(Q,1:Precision:=10,
 BadPrimes:=[<3,c,1>])): c in [6..10]];
```

The returned number for $c = 9$ is 0.0000000000, while the numbers for the other c are all at least 0.1. This information strongly suggests that indeed $F_3(M,x) = 1$ and $c_3 = 9$. After setting up `L` with `[<3,9,1>]`, analytic calculations can be done as before. For example, here the order of central vanishing is apparently 2. In the miraculous command `CFENew`, `CFE` stands for the Magma command `CheckFunctionalEquation`, implemented by Tim Dokchitser using his [Dok04]; `New` reflects subsequent improvements by Watkins.

13 Bad primes

Fix a hypergeometric motive $M = H(Q,t)$ and a prime p, which is tame or wild for (Q,t) as defined in Section 10. We now sketch how Magma computes the local invariants when p is tame, and describe some conjectural basic features when p is wild.

Tame primes. When p is tame for (Q,t), t agrees modulo p with one of the three cusps $\tau \in \{0,1,\infty\}$. The conductor exponent c_p is the codimension of the invariants of a power of the corresponding Levelt matrix h_τ from Section 2. When $\tau = 1$, the simple shape of h_1 from Section 9 gives a completely explicit formula: $c_p = 1$ except in the orthogonal case with $\mathrm{ord}_p(t-1)$ even, where $c_p = 0$. When $\tau \in \{0,\infty\}$,

$$c_p = \mathrm{rank}(h_\tau^{|k|} - I). \tag{13.1}$$

Here $k = \mathrm{ord}_p(t)$, $\tau = \infty$ if k is negative, and $\tau = 0$ if k is positive. So there is separate periodic behavior for $k < 0$ and $k > 0$, as illustrated by the top part of Figure 13.1. The example of this table comes from the case $(a,b) = (3,5)$ of (5.1), so the conductor there is very simply computed as the discriminant of the octic algebra $\mathbb{Q}[x]/(5x^8 + 8tx^5 + 3t^3)$.

Because ramification is at worst tame, the degree of $F_p(M,x)$ is $n - c_p$. When $\tau = 1$, Magma computes $F_p(M,x)$ by slightly modifying the formulas for point counts sketched in Section 10. In the other cases, Magma computes $F_p(M,x)$ from how the family $X_{Q,t}$ degenerates at the relevant cusp $\tau \in \{0,\infty\}$.

Wild primes. To simplify the overview, we just exclude the case where $\mathrm{ord}_p(t-1) \geq 1$. Write specialization points as $t = vp^k$ with $k = \mathrm{ord}_p(t)$. The bottom part of Figure 13.1 shows right away that the situation is complicated.

A function σ is graphed in both parts of Figure 13.1 and its general definition goes as follows. For d a positive integer, write

$$s(d) = \begin{cases} 1 & \text{if } \gcd(d,p) = 1, \\ 1 + \mathrm{ord}_p(d) + \frac{1}{p-1} & \text{else.} \end{cases}$$

Let

$$\sigma_\infty = \sum_{i=1}^n s(\mathrm{denom}(\alpha_i)), \qquad \sigma_0 = \sum_{i=1}^n s(\mathrm{denom}(\beta_i)).$$

Define $k_{\mathrm{crit}} = \sigma_\infty - \sigma_0 = -\sum_j \gamma_j \mathrm{ord}_p(\gamma_j)$ and transition points $k_\infty = \min(k_{\mathrm{crit}}, 0)$ and $k_0 = \max(k_{\mathrm{crit}}, 0)$. Then

$$\sigma(k) = \begin{cases} \sigma_\infty & \text{if } k \leq k_\infty, \\ \max(\sigma_\infty, \sigma_0) - |k| & \text{if } k_\infty \leq k \leq k_0, \\ \sigma_0 & \text{if } k \geq k_0. \end{cases}$$

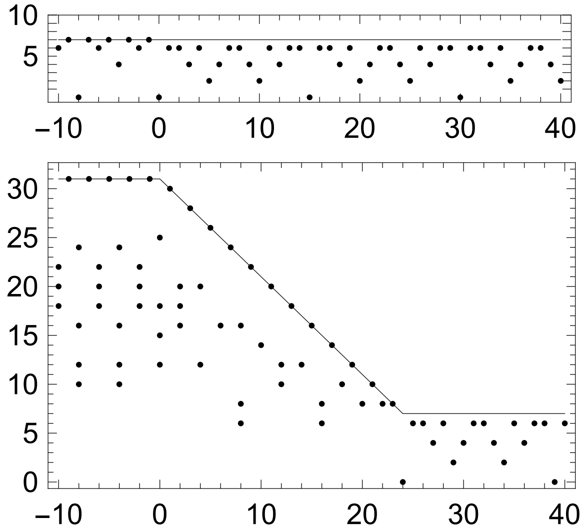

Figure 13.1: Pairs (k, c_p) where $k = \mathrm{ord}_p(t)$ and $c_p = \mathrm{ord}_p(\mathrm{Conductor}(H([-8, 3, 5], t)))$, compared with the graph of the corresponding σ. Top: The tame cases $p > 5$. Bottom: The wild case $p = 2$.

In the tame case, σ is just the constant function n. In general, there are plateaus corresponding to the cusps ∞ and 0, and then a ramp of length $|k_{\mathrm{crit}}|$ between them.

We conjecture that

$$c_p \leq \sigma(k) - \mathrm{degree}(F_p(M, x)), \qquad (13.2)$$

with equality if k and p are relatively prime. The second statement is proved in [LNV84] in the general trinomial setting of (5.1). Outside this setting, (13.2) has been computationally verified in many instances. As one passes from one family to another via mod ℓ congruences as in Section 11, wild ramification at p does not change. This fact and other theoretical stabilities give us confidence in (13.2). To make Magma fully automatic, one would have to determine the more complicated function $\sigma(k, v)$ with $c_p = \sigma(k, v) - \mathrm{degree}(F_p(M, x))$.

At present, we understand a factor $f_p(M, x)$ of the Frobenius polynomial $F_p(M, x)$ as follows. For $k \neq k_{\mathrm{crit}}$, $f_p(M, x)$ comes from modifying the tame formulas; in particular its degree is given by replacing k with $k - k_{\mathrm{crit}}$ in (13.1). If $k = k_{\mathrm{crit}}$, corresponding to being at the bottom of the ramp, we use an *erasing* principle explained to us by Katz. Here one simply ignores all α_i and β_j that have denominator divisible by p. Let n_∞ and n_0 be respectively the number of α_i's and β_j's remaining. Then $n_\infty - n_0$ is a multiple of $p - 1$, so that the formulas of Section 10 still make sense, as the choice of an additive character on $\mathbb{F}_p$ again does not matter. The resulting $f_p(M, x)$ has degree $\max(n_\infty, n_0)$. We conjecture that the complementary factor $F_p(M, x)/f_p(M, x)$ is 1 whenever p and k are relatively prime. In practice, when $p \mid k$ it is usually 1 also, but not always.

14 Automorphy

One of the most exciting aspects of the theory of motives is its conjectural connection to the theory of automorphic forms through the Langlands program. It is conjectured that for every motive M there is an automorphic form f with

$$L(M, s) = L(f, s). \qquad (14.1)$$

Automorphic L-functions $L(f, s)$ are known to have the analytic continuation and functional equation conjectured of motivic L-functions. HGMs form a large class of motives in which one can see proved instances of this correspondence, and numerically seek more instances.

The case $n = 2$. Suppose $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ has nonvanishing Hodge numbers $h^{w,0} = h^{0,w} = 1$ and conductor N. Then there is a corresponding power series $f \in \mathbb{Z}[[q]]$. Via $q = e^{2\pi i z}$, it can be viewed as a "newform" on the group $\Gamma_0(N)$, as in (1.6), except now the modular weight is $w + 1$.

To see some explicit f fitting into proved instances of (14.1) beyond the setting $w = 1$ of elliptic curves, consider the four reflexive parameters Q yielding motives $H(Q, 1)$ with Hodge vector $(1, 1, 0, 0, 1, 1)$:

Q	a_5	a_7	N'		b_5	b_7	N''	
Φ_2^6/Φ_1^6	-2	24	8	$\eta_2^4\eta_4^4$	-74	-24	8	
$\Phi_2^4\Phi_3/\Phi_1^4\Phi_6$	-18	8	12		54	-88	4	η_2^{12}
$\Phi_2^2\Phi_3^2/\Phi_1^2\Phi_6^2$	-6	-16	18		-66	176	6	
Φ_3^3/Φ_6^3	-16	-12	72		-16	12	72	

Magma computes automatically with these reducible motives, reporting their conductors to be $N = 64$, 48, 108, and 5184. However these computations do not see the decompositions $H(Q, 1) = M'(-1) \oplus M''$ analogous to (9.1), where now M' and M'' respectively have Hodge vectors $(1, 0, 0, 1)$ and $(1, 0, 0, 0, 0, 1)$. In the Frobenius polynomial

$$F_p(H(Q, 1), x) = (1 - pa_p x + p^5 x^2)(1 - b_p x + p^5 x^2),$$

the pa_p belonging to $M'(-1)$ can be distinguished from the b_p belonging to M'' whenever the latter is not a multiple of p. The reader might enjoy searching in the LMFDB's complete lists [LMF21] of newforms to see that the a_p and b_p just for $p = \{5, 7\}$ are sufficient to identify the relevant forms and in particular determine the above-displayed factorizations $N = N'N''$. Part of the further information given by the LMFDB is that two of the forms are expressible using the Dedekind eta function η_1, via $\eta_d := q^{d/24} \prod_{j=1}^{\infty}(1 - q^{dj})$.

Higher rank. The motivic Galois group G_M of a given motive M determines what type of automorphic form should match it, and other invariants of M say more precisely where to look. As an example, [DPVZ20] considers orthogonal motives with Hodge vector $(1, 1, 0, 1, 1)$ and finds matching

Hilbert modular forms of modular weight $(2,4)$. Generally speaking, the Hodge numbers of central concern earlier in this survey continue to play a large role. In particular, motives for which all $h^{p,q}$ are 0 or 1 have theoretical advantages, and their L-functions at least have a meromorphic continuation with the right functional equation [PT15].

15 Numerical computations

We promised in the introduction that we would equip the reader to numerically explore a large collection of motivic L-functions. We conclude this survey by giving sample computations in the context of two important topics, always assuming that the expected analytic continuation and functional equation indeed hold. In both topics, we let $c = \frac{1}{2} + \frac{w}{2}$ be the center of the functional equation. The conductors N in our examples are small for their Hodge vectors h, allowing us to keep runtimes short and/or work to high precision.

Special values. If M is a motive in $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ then the numbers $L(M,k)$ for integers $k \leq c$ are mostly forced to be 0, because of poles in the infinity factor (12.2) and the functional equation. However, when $L(M,k)$ is nonzero it is expected to be arithmetically significant [Del79]. The arithmetic interpretation involves a determinant of periods like (2.1). To see the significance without entering into periods, one can look at the ratio $r_d = (\sqrt{d}/4)^e L(M \otimes \chi_d, k)/L(M,k)$, for d a positive quadratic discriminant. Here the normalizing exponent e depends on the Hodge numbers of M via $e = \sum_{p>q} h^{p,q}(p-q)$. The periods then cancel out so that r_d should be rational.

For a sample computation, take $M = H(\Phi_2^5/\Phi_1^5, 2^{10})$ so that the Hodge vector is $h = (1,1,1,1,1)$ and $e = 6$. Use (2.4) and `L:=LSeries(Q,1024)` to define its L-function, as usual. While 2 is wild for the family, it is unramified in M because the exponent 10 is at the bottom of the ramp of Section 13. The erasing procedure from the end of that section applies, yielding

$$F_2(M,x) = (1-4x)(1+5x+10x^2+80x^3+256x^4).$$

Since $t - 1 = 1023 = 3 \cdot 11 \cdot 31$ is squarefree, it is the conductor, by the recipe before (13.1). Magma gets all the bad factors right automatically. As a confirmation, `CFENew(L)` quickly returns 0 to the default 30 digits.

`Evaluate(L,2)` gives $0.4278180899\cdots$. Twisting by a d with $\gcd(d,1023) = 1$ makes the conductor go up by a factor of d^5 and precision needs to be reduced.

```
Evaluate(LSeries(Q,1024:
QuadraticTwist:=5,Precision:=10),2);
```

takes six minutes to give its answer of 35.04685793. The normalized ratio r_5 and then two others are apparently

$$r_5 = \frac{5}{2}, \qquad r_8 = \frac{5}{2^3}, \qquad r_{13} = \frac{13 \cdot 251}{2^2}.$$

The two L-functions appearing in r_d are completely different analytically, and so the apparent fact that the quotients r_d are rational is very remarkable.

Readers wanting to work out their own examples might want to begin with M having odd weight and evaluating L-functions at the central point c, where one expects a generalization of the conjecture of Birch and Swinnerton-Dyer to hold. A model in this context is $H(\Phi_2^4/\Phi_1^4, -2^8)$ with Hodge vector $h = (1,1,1,1)$ and thus $w = 3$, $c = 2$, and $e = 4$. Here for d prime to 257, the conductor of the twisted L-function is $257d^4$. The ratio r_d should be zero if d is not a square modulo 257. Computations become progressively more difficult, but for $d \leq 41$, eight zeros are seen this way, with $r_d \in \{1,4\}$ otherwise.

Critical zeros. For a weight w motive M, all the zeros of the completed L-function $\Lambda(M,s)$ lie in the critical strip $c - \frac{1}{2} \leq \mathrm{Re}(s) \leq c + \frac{1}{2}$. The Riemann hypothesis for M then predicts that all the zeros lie on the critical line $\mathrm{Re}(s) = c$. We now show by examples that numerical identification of low-lying zeros is possible in modestly high rank.

For rank ten examples, take $M_{10,w} = H(Q_w, 1)$ with $Q_{10} = \Phi_4\Phi_2^9/\Phi_1^{11}$ and $Q_7 = \Phi_4^4\Phi_2^4/\Phi_8^2\Phi_1^4$. So $M_{10,10}$ is orthogonal with Hodge vector $(1,1,1,1,1,0,1,1,1,1,1)$ while $M_{10,7}$ is symplectic with Hodge vector $(1,1,2,1,1,2,1,1)$.

The only bad prime in each case is 2. A search says that $F_2(M_{10,10},x) = 1+32x$ and $c_2 = 11$. For $M_{10,7}$, $k = k_{\mathrm{crit}} = 0$ so erasing applies, yielding $1+4x+96x^2+512x^3+16384x^4$ as a factor of $F_2(M_{10,7},x)$. A short search says that this factor is all of $F_2(M_{10,7},x)$ and $c_2 = 18$.

In general, the Hardy Z-function of a motive M is

$$Z(M,t) = \epsilon^{1/2} \frac{N^{s/2}\Gamma_{h,\sigma}(s)}{|N^{s/2}\Gamma_{h,\sigma}(s)|} L(M,s),$$

with $s = c+it$. It is a real-valued function of the real variable t, even or odd depending on whether the sign ϵ is 1 or -1.

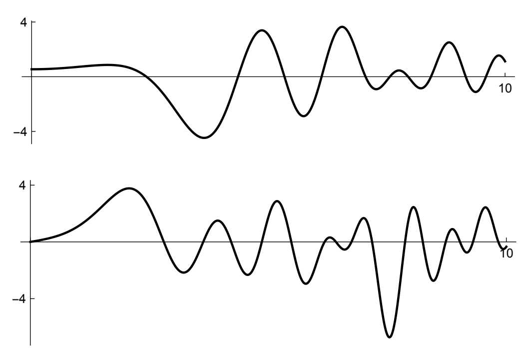

Figure 15.1: Graphs of $Z(M_{10,10},t)$ and $Z(M_{10,7},t)$

Figure 15.1 was computed via many calls to `Evaluate` at points of the form $c+it$. The signs in the two cases are 1 and -1, and the orders of central vanishing are the minimum possible, 0 and 1. For the three cases in the opening picture, the orders of central vanishing are respectively 2, 1, and 2. On all five plots, all local maxima are above the axis and all local minima are beneath the axis. Zeros off the critical line would likely cause a disruption of this pattern, Thus the plots not only identify zeros on the critical line, but suggest a lack of zeros off the critical line.

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