Explicit Plancherel measures for counting L-functions

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Overview

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}$$

is a remarkable number-theoretic object. Its completion

$$\Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

is entire except for poles at 0 and 1, has a functional equation $\Lambda(s) = \Lambda(1 - s)$, and its zeros conjecturally are all on $\mathrm{Re}(s) = 1/2$.

The Riemann zeta is the first element of a countable set $\mathcal{L}$ of standard automorphic L-functions. It is a member of a subset $\mathcal{L}_0$ of L-functions which look like they come from algebraic geometry. Millions of L-functions $L \in \mathcal{L}$, most in $\mathcal{L}_0$, are cataloged on the LMFDB. **In this talk we explain how Plancherel measures are useful for approximating cardinalities of parts of $\mathcal{L}$ and understanding how $\mathcal{L}_0$ sits inside $\mathcal{L}$.**

Sections of today's talk

1. General setting: Automorphic representations $\rightarrow \mathcal{L} \supset \mathcal{L}_0 \leftarrow$ motives

2. Dot diagrams for infinity factors of L-functions

3. Tour of the LMFDB, guided by infinity factors

4. Plancherel densities governing distribution of infinity factors

5. Two classes of infinity factors not in the LMFDB.

1. General setting

Let n be a positive integer. An L-function of degree n for this talk is a Dirichlet series with Euler product coming from a "balanced ∞-tempered cuspidal automorphic representation π of the adelic group $GL_n(\mathbb{A})$". An L-function has the form

$$L(s,\pi) = \sum_{k=1}^{\infty} \frac{a_k}{k^s} = \prod_p \frac{1}{f_p(p^{-s})}$$

with $f_p(x) = 1 - a_p x + \cdots$ a polynomial in $\mathbb{C}[x]$ of degree $\leq n$.
An L-function comes with a *conductor*

$$N \geq 1.$$

An L-function also comes with a decomposition of its degree, $n = r_1 + 2r_2$, and two unordered lists of complex numbers called *spectral parameters*:

$$\mu_1, \ldots, \mu_{r_1}, \text{ and } \nu_1, \ldots, \nu_{r_2}.$$

The balanced condition is that $\mathrm{Im}(\sum \mu_k + 2\sum \nu_k) = 0$. The ∞-tempered condition is that always $\mathrm{Re}(\mu_k) \in \{0, 1\}$ and $\mathrm{Re}(\nu_k) \in \{1/2, 1, 3/2, 2, \ldots\}$.

Completed L-functions: AC and FE

Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and let $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then the *infinity factor* is

$$L_\infty(s) = \prod_{k=1}^{r_1} \Gamma_{\mathbb{R}}(s+\mu_k) \prod_{k=1}^{r_2} \Gamma_{\mathbb{C}}(s+\nu_k).$$

The completed L-function is

$$\Lambda(s,\pi) = N^{s/2} L_\infty(s) L(s,\pi).$$

Outside the Riemann zeta function case, where there are poles at $s=0$ and $s=1$ only, $\Lambda(s,\pi)$ has an *analytic continuation* to the whole s plane.

The complex conjugate representation $\overline{\pi}$ gives the complex conjugate L-function $L(s,\overline{\pi}) = \sum_k \overline{a}_n n^{-s}$ with auxiliary data N, $\overline{\mu}_k$, and $\overline{\nu}_k$. One has the *functional equation* $\Lambda(1-s,\pi) = \epsilon\Lambda(s,\overline{\pi})$ for some number ϵ on the unit circle. The self-dual case $\pi = \overline{\pi}$ is particularly important and breaks into the *symplectic* and *orthogonal* subcases.

Definition of $\mathcal{L}$ and $\mathcal{L}_0$

Definition

- *$\mathcal{L}$ is the set of all L-functions.*
- *$\mathcal{L}_0$ is the subset of L-functions with totally real infinity factors, i.e. those for which the spectral parameters μ_k and ν_k are all real.*

Some L-functions in $\mathcal{L}_0$ come from geometry and are called *motivic*. Motivic L-functions have many further remarkable properties, beginning with $a_n \in \overline{\mathbb{Q}}$.

The other L-functions are less tangible and (for the moment at least!) viewed as less important. They are called *transcendental*.

It is conjectured that motivic L-functions are exactly the ones in $\mathcal{L}_0$. All this leads to . . .

A Huge Community Project

Explicitly describe $\mathcal{L}$, paying special attention to $\mathcal{L}_0$.

2. Dot diagrams for infinity factors

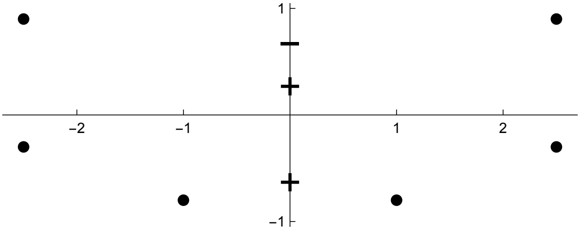
To give a degree n infinity factor in the usual way is the same as to give n dots in $\frac{1}{2}\mathbb{Z} \times i\mathbb{R} \subset \mathbb{C}$ with real sum, stable under reflection in the imaginary axis, together with a sign attached to each purely imaginary dot, via

$$\Gamma_{\mathbb{R}}(s+it) \leftrightarrow [it_+], \quad \Gamma_{\mathbb{R}}(s+1+it) \leftrightarrow [it_-], \quad \Gamma_{\mathbb{C}}(s+\nu) \leftrightarrow [\nu]+[-\bar{\nu}].$$

For example, the infinity factor

$$\Gamma_{\mathbb{R}}(s-0.6i)\Gamma_{\mathbb{R}}(s+0.3i)\Gamma_{\mathbb{R}}(s+1+0.7i)\Gamma_{\mathbb{C}}(s+1-0.8i)\Gamma_{\mathbb{C}}(s+\frac{5}{2}-0.3i)\Gamma_{\mathbb{C}}(s+\frac{5}{2}+0.9i)$$

has dot diagram



A dot diagram is self-dual iff it is also symmetric with w.r.t. the real axis.

Spaces of infinity factors

Let X be the space of infinity factors. Connected components of this space are indexed by multiplicity vectors of real parts:

$$h = \left(\ldots, h^{-3/2}, h^{-1}, h^{-1/2}, \begin{matrix} h^{0+} \\ h^{0-} \end{matrix}, h^{1/2}, h^{1}, h^{3/2}, \ldots \right).$$

We write H for the set of such h and define subsets H^+ and H^- via

- $h \in H^+$ if h^j is *even* for all half-integers j.
- $h \in H^-$ if h^j is *even* for all integers j and also $j \in \{0+, 0-\}$.

Let X^+ and X^- be the subspaces of X consisting of infinity factors which can arise from orthogonal and symplectic L-functions respectively. Then the decompositions into connected components take the form

$$X = \coprod_{h \in H} X(h), \qquad X^\tau = \coprod_{h \in H^\tau} X^{\mathrm{sd}}(h).$$

When $h \in H^\tau$ we also write $X^{\mathrm{sd}}(h)$ as $X^\tau(h)$.

The above notation is imported from algebraic geometry, where Hodge numbers $h^{p,q}$ are now written $h^{(p-q)/2}$ and there is a natural decomposition $h^0 = h^{0+} + h^{0-}$ coming from a complex conjugation.

Dimensions and real points

The dimension of a connected component is

$$\dim X(h) = -1 + h^{0+} + h^{0-} + \sum_{j>0} h^j, \quad \dim X^{\mathrm{sd}}(h) = \lfloor \tfrac{h^{0+}}{2} \rfloor + \lfloor \tfrac{h^{0-}}{2} \rfloor + \sum_{j>0} \lfloor \tfrac{h^j}{2} \rfloor.$$

These formulas are clear from the dot diagrams. For example, in the self-dual case the number given is the number of dots $x + iy$ with $x \geq 0$ and $y > 0$ in the case where all dots are different.

Note that every component of either X or X^{sd} contains exactly one totally real point. We say that h has *pure parity* if all the j with $h^j > 0$ have the same parity, meaning integral or half-integral. In the pure-parity case, this real point is the only one can that can arise from algebraic geometry. In the complementary *mixed parity* case, no points can arise from algebraic geometry and so one expects that the real point does not actually come from an L-function at all. Note that $\dim X^{\tau}(h) = 0$ exactly when $h^j \leq 1$ always, and this implies that h has pure parity.

3. Guided (by infinity factors) tour of the LMFDB

Some dot diagrams of infinity factors:

n	In LMFDB		Not in LMFDB	
1	$+$	$-$		
2	$\bullet \quad \bullet$	ϵ ϵ	$+$ $-$	
3	$+$ $+$ $+$	$\bullet \quad + \quad \bullet$	$\bullet \qquad \bullet$ ϵ	$-$ $-$ $+$
4	$+$ $+$ $+$ $+$	$\bullet \quad \bullet \quad \bullet \quad \bullet$	$\bullet \quad \bullet$ $\bullet \quad \bullet$	ϵ $\bullet \qquad \bullet$ ϵ

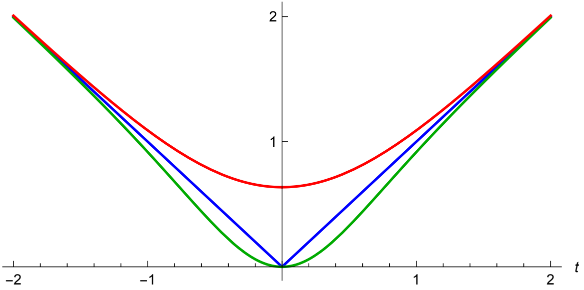
Examples for the two blue cases will be given in Section 5.

4. Plancherel densities and the equidistribution principle

Let $|j + it|_0$ be the usual absolute value on $\mathbb{C}$, restricted to $\frac{1}{2}\mathbb{Z} + i\mathbb{R}$. For purely imaginary numbers use also the variants

$$|it|_+ = t \tanh\left(\frac{\pi}{2}t\right), \qquad |it|_- = t \coth\left(\frac{\pi}{2}t\right),$$

Picture of the standard absolute value $|it|_0$ and its variants $|it|_+$ and $|it|_-$:



For $|t|$ even slightly large, all three $|\cdot|_\tau$ are approximately the same.

Fix a type $\tau \in \{0, +, -\}$ with $0 =$ non-self-dual, $+ =$ orthogonal, and $- =$ symplectic. Then for typed points α_a, β_b, define

$$\epsilon(\tau, \alpha_a, \beta_b) = \begin{cases} \tau(-1)^{1+2\,\mathrm{Re}(\alpha)} & \text{if } \alpha = \bar{\beta} \text{ and } \mathrm{Re}(\alpha) \neq 0, \\ a\,b & \text{else} \end{cases} \quad \text{in } \{0, +, -\}.$$

Main Principle

For a fixed h and conductor N, infinity factors of L-functions of type h are distributed in $X^\tau(h)$ *proportionally to Plancherel density:*

$$\begin{aligned} \Delta_0(\alpha) &:= \prod_{j<k} |\alpha_j - \alpha_k|_{\epsilon(0,\alpha_j,\alpha_k)} \\ \Delta_+(\alpha) &:= \prod_{j<k} |\alpha_j - \alpha_k|_{\epsilon(+,\alpha_j,\alpha_k)}^{(1-\delta_{\alpha_j,-\alpha_k})/2} \\ \Delta_-(\alpha) &:= \prod_{j<k} |\alpha_j - \alpha_k|_{\epsilon(-,\alpha_j,\alpha_k)}^{(1+\delta_{\alpha_j,-\alpha_k})/2} \end{aligned}$$

So Plancherel densities can be thought of as modified discriminants. As one runs over all h's belonging to a fixed degree n, there is also some uniformity in the proportionality factor.

Intuitive reformulation

For fixed h, one can think of points sliding up and down on vertical lines. Always left-right symmetry is imposed. In the orthogonal and symplectic cases, an up-down symmetry is imposed too. The formulas say roughly that the further away the points are from each other, the more likely a configuration is to arise from an L-function.

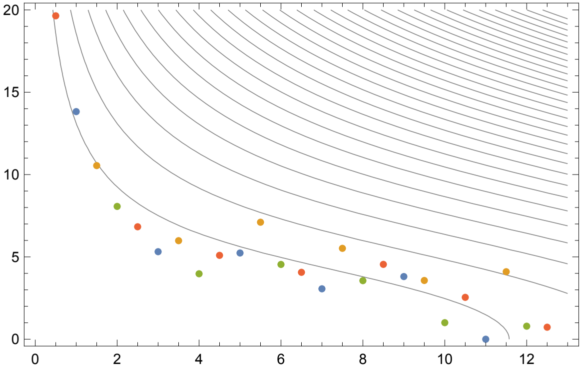
One can think in terms of almost every pair of points $\{\alpha_i, \alpha_j\}$ repelling each other in the same fashion, with force inversely proportional to $|\alpha_i - \alpha_j|$. This force corresponds to the factor $|\alpha_i - \alpha_j|$ in the non-self-dual case and to the factor $|\alpha_i - \alpha_j|^{1/2}$ in the self-dual cases. For pairs of points on the same vertical line one sometimes needs a modification which is particularly important for close-range interactions:

- Points of $\left\{\begin{array}{l}\text{opposite}\\ \text{the same}\end{array}\right.$ sign repel each other $\left\{\begin{array}{l}\text{less}\\ \text{more}\end{array}\right.$.
- Conjugate points on $\left\{\begin{array}{l}\text{expected}\\ \text{unexpected}\end{array}\right.$ lines repel each other $\left\{\begin{array}{l}\text{less}\\ \text{more}\end{array}\right.$.

In the $\left\{\begin{array}{l}\text{orthogonal}\\ \text{symplectic}\end{array}\right.$ case the force is $\left\{\begin{array}{l}\text{set to zero}\\ \text{doubled}\end{array}\right.$ for opposite points.

5. Two classes of infinity factors not in the LMFDB

For *rank three L-functions* with $L_\infty(s) = \Gamma_\mathbb{R}(\delta_{j,\mathbb{Z}} - 2it + s)\Gamma_\mathbb{C}(j + it + s)$ with $t \geq 0$ the density function is $2j(j^2 + 9t^2)$ is contour-plotted:

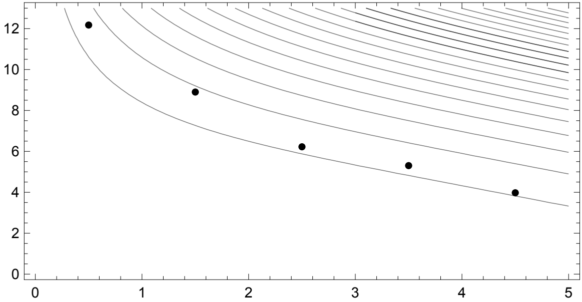


For each $j = 1/2,\ 1,\ 3/2, \ldots$ the smallest t arising from an L-function with $N = 1$ is indicated by a dot. The dot at $j + it = 11$ is the symmetric square of the Ramanujan L-function; all others are transcendental.

For *rank four symplectic L-functions* with

$$L_\infty(s) = \Gamma_{\mathbb{C}}(j + it + s)\Gamma_{\mathbb{C}}(j - it + s),$$

the density function $16(j^2 + t^2)j(t\coth(\pi t/2))$ is contour-plotted in the j-t plane:



For each $j = 1/2$, $3/2$, $5/2$, $7/2$, $9/2$, the smallest t arising from an L-function with $N = 1$ is indicated by a dot.