Hurwitz-Belyi maps

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Overview

1. **Background on Belyi maps via an unusual example with**
   - **degree 64 and monodromy group all of S_{64},**
   - **field of definition $\mathbb{Q}$, and**
   - **bad reduction set only $\{2, 3\}$.**

2. **A conjecture on the existence of certain Belyi maps in arbitrarily large degree.**

3. **Hurwitz-Belyi maps: definition, numerical examples, and how perhaps they can prove the conjecture.**

Parts 2 and 3 are analogs for Belyi maps of work with Akshay Venkatesh on number fields.

The Belyi map case is similar to the number field case but more geometric.

1A. Generic vs. Belyi maps

Any degree n function $F : \mathrm{Y} \to \mathrm{P}^1$ has

$$2n + 2\ \mathrm{genus}(Y) - 2$$

critical points in Y, counting multiplicity. All explicit examples today will have $\mathrm{genus}(Y) = 0$. However the theoretical considerations require general genus.

Generically, the critical points $y_i \in \mathrm{Y}$ are distinct; also the critical values $F(y_i) \in \mathrm{P}^1$ are distinct.

Definition. *F is called a Belyi map if its critical values are within $\{0, 1, \infty\}$.*

So Belyi maps are as far from generic as possible, and moreover their critical values are normalized.

1B. The unusual Belyi map

Define $\beta : \mathrm{P}^1_y \to \mathrm{P}^1_t$ by

$$\beta(y) = \frac{(y+2)^9 y^{18} (y^2-2)^{18} (y-2)}{(y+1)^{16} (y^3-3y+1)^{16}}.$$

so that $F(\infty) = 1$. Where are the

$$2n + 2g - 2 = 2 \cdot 64 + 2 \cdot 0 - 2 = 126$$

critical points?

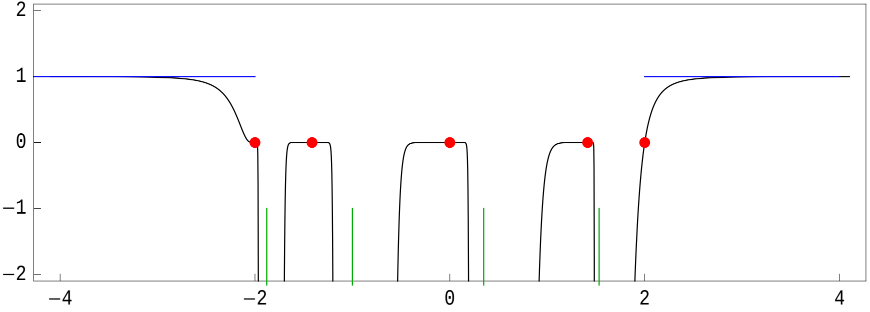
0: From numerator, $8 + 3 \cdot 17 = \boxed{59}$ crit points with crit value 0.

∞: From denominator, $4 \cdot 15 = \boxed{60}$ crit points with crit value ∞.

1: From degree(numerator − denominator) $= 56$, ∞ is a crit point with multiplicity $\boxed{7}$.

Since $59 + 60 + 7 = 126$, β is indeed a Belyi map.

1C. A real-variable visualization of β



Clearly visible:

0: Zeros with multiplicity 9, 18, 18, 18, 1.

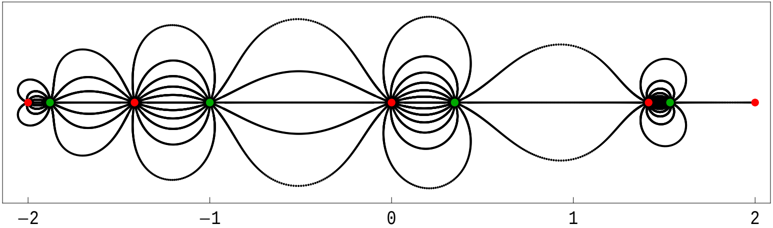
1: Horizontal asymptote at level $t = 1$ with multiplicity 8.

∞: Poles with multiplicity 16, 16, 16, 16.

Also there are 56 non-critical non-real preimages of 1. This makes the ramification triple $(\lambda_0, \lambda_1, \lambda_\infty) = (18^3\, 9\, 1,\ 8\, 1^{56}, 16^4)$.

1D. A complex-variable visualization of β

The *dessin* $\beta^{-1}([-\infty, 0]) \subset \mathrm{P}^1_y$:



Monodromy operators on the set of edges can be read off:

g_∞ = rotate counterclockwise minimally about $\bullet$.
g_0 = rotate counterclockwise minimally about $\bullet$.
$g_1 := {g_0}^{-1} {g_\infty}^{-1}$ so that $g_0 g_1 g_\infty = 1$.

Each g_i has cycle type λ_i and $\langle g_0, g_1, g_\infty \rangle = S_{64}$.

1E. Computation of β

To obtain β, first consider

$$\frac{(y+2)^9\left(y^3+ay^2+by+c\right)^{18}(y-2)}{\left(y^4+dy^3+ey^2+fy+g\right)^{16}}$$

There are seven equations in the seven unknowns a, b, c, d, e, f, g.

It turns out there are 35 solutions (a, b, c, d, e, f, g), and these correspond to 35 Belyi maps. The a-values are the roots of

$$a\left(8096790625a^{34} - 1360260825000a^{33} + \cdots \right.$$
$$\left. -633054568549175937272241607139131392000\right).$$

The first factor a gives our β via the solution

$$(a, b, c, d, e, f, g) = (0, -2, 0, 1, -3, -2, 1).$$

The second factor has Galois group S_{34} and field discriminant

$$2^{71}3^{44}5^{27}7^{27}11^{23}13^{19}19^{15}23^{10}29^{11}31^{8}37^{4}47^{3}.$$

1F. Three invariants of a degree n Belyi map

(illustrated by our β with comments)

- The monodromy group $\Gamma \subseteq S_n$. For β, it's $\boxed{S_{64}}$. It's very easy to make Γ *full*, meaning $\Gamma \in \{A_n, S_n\}$.

- The field of definition $F \subset \mathbb{C}$. For our β, it's $\boxed{\mathbb{Q}}$, despite

$$\mathrm{degree}(18^3\,9\,1,\ 8\,1^{56}, 16^4) = 35.$$

When $\mathrm{degree}(\lambda_0, \lambda_1, \lambda_\infty)$ is large, very commonly all of the corresponding Belyi maps are conjugate.

- The bad reduction set $\mathcal{P}$. For β, it's just $\boxed{\{2, 3\}}$ (because for other p the numerator and denominator are coprime in $\mathbb{F}_p[y]$). $\mathcal{P}$ is a subset of the primes dividing $|\Gamma|$. So when $\Gamma = S_n$, this bound is all primes $\leq n$. Typically $\mathcal{P}$ is close to or equal to its upper bound.

2A. A personal timeline

In 2004-2005, Gunter Malle and I tried to find full Belyi maps defined over $\mathbb{Q}$ and ramified within $\{2, 3\}$ for as many degrees as possible. We found them only in degrees (1, 2, 3, 4, 6), 9, 10, 12, 18, 28, and 33. We specialized them to number fields with discriminant $\pm 2^a 3^b$.

In 2007-2008, I built some unusual rational functions $T_{m,n}$ and $U_{m,n}$ from Chebyshev polynomials. The map $T_{8,9}$ and the above-described $U_{8,9}$ add 36 and 64 to the list. I also explained how Bhargava's heuristic approximation to the number of number fields with a given discriminant suggests that *there should be only finitely many full Belyi maps defined over $\mathbb{Q}$ and ramified within a given finite set of primes $\mathcal{P}$.*

In 2012-2015, Akshay Venkatesh and I worked out how Belyi maps constructed from specializing Hurwitz covers completely escape the Bhargava heuristic. In particular, they almost definitely render the italicized statement false for some $\mathcal{P}$.

2B. A weird distinction

Definition

Call a finite set of primes $\mathcal{P}$ *anabelian* if it contains the set of primes dividing the order of a finite nonabelian simple group, and *abelian* otherwise.

Examples. The set $\mathcal{P} = \{2, 3, p\}$ is anabelian for $p \in \{5, 7, 13, 17\}$ because of the following simple groups:

p	G	$\lvert G\rvert$			p	G	$\lvert G\rvert$		
5	A_5	60	$=$	$2^2 3^1 5$	7	$SL_3(2)$	168	$=$	$2^3 3^1 7$
5	A_6	360	$=$	$2^3 3^2 5$	7	$SL_2(8)$	504	$=$	$2^3 3^2 7$
5	$W(E_6)^+$	25920	$=$	$2^5 3^4 5$	7	$SU_3(3)$	6048	$=$	$2^5 3^3 7$
13	$SL_3(3)$	5616	$=$	$2^4 3^3 13$	17	$PSL_2(17)$	2448	$=$	$2^4 3^2 17$

All other $\mathcal{P}$ with $|\mathcal{P}| \leq 3$ are abelian, by the classification of finite simple groups. Also if $2 \notin \mathcal{P}$, then $\mathcal{P}$ is abelian.

The refined expectations

Conjecture

Let $\mathcal{P}$ be an anabelian set of primes (like $\{2,3,5\}$ or $\{2,3,7\}$). Then there exist $\boxed{\textit{full}}$ *Belyi maps,* $\boxed{\text{defined over } \mathbb{Q}}$*, and* $\boxed{\text{ramified within } \mathcal{P}}$*, of arbitrarily large degree n.*

Strong reason for believing: the existence of Hurwitz-Belyi maps.

Personal Guess

Let $\mathcal{P}$ be an abelian set of primes (like $\{2,3\}$ or $\{3,5,7,11,13\}$). Then there exist $\boxed{\textit{full}}$ *Belyi maps,* $\boxed{\text{defined over } \mathbb{Q}}$*, and* $\boxed{\text{ramified within } \mathcal{P}}$*, only for finitely many degrees n.*

Weak reason for believing: all examples with n large for a given $\mathcal{P}$, like β from before, seem "accidental".

3A. Hurwitz parameters

Consider again a general degree n map $F : \mathsf{Y} \to \mathsf{P}^1_t$. Three invariants are:

- Its global monodromy group $G \subseteq S_n$
- The list $C = (C_1, \ldots, C_s)$ of distinct conjugacy classes arising as non-identity local monodromy operators.
- The corresponding list $(D_1, \ldots, D_s)$ of disjoint finite subsets $D_i \subset \mathsf{P}^1_t$ over which these classes arise.

To obtain a single discrete invariant, we write $\nu = (\nu_1, \ldots, \nu_s)$ with $\nu_i = |D_i|$. We then form the *Hurwitz parameter* $h = (G, C, \nu)$.

Let $r := \sum |\nu_i|$. The case $r = 3$ with $\nu = (1, 1, 1)$ is the standard context for Belyi maps, with (D_1, D_2, D_3) normalized to $(\{0\}, \{1\}, \{\infty\})$. We will extract particularly interesting Belyi maps from the cases $r \geq 4$.

3B. Hurwitz maps

An r-point Hurwitz parameter $h = (G, C, \nu)$ determines a cover of r-dimensional complex varieties:

$$\pi_h : \mathrm{Hur}_h \to \mathrm{Conf}_\nu.$$

A point $x \in \mathrm{Hur}_h$ indexes an isomorphism class of covers

$$\mathsf{Y}_x \to \mathsf{P}^1_t \tag{1}$$

of type h. The base Conf_ν is the space of possible branch loci

$$(D_1, \ldots, D_s). \tag{2}$$

The map π_h sends a cover (1) to its branch locus (2).
When G is simple and C contains sufficiently many classes,

$$\mathrm{degree}(\pi_h) = \frac{1}{|G|^2} \prod_{i=1}^{s} |C_i|^{\nu_i}.$$

3C. Known facts about Hurwitz maps

Monodromy: The main theorem with Venkatesh gives necessary and sufficient conditions on a pair (G, C) for $\pi_h : \mathrm{Hur}_h \to \mathrm{Conf}_\nu$ to have $\boxed{\text{full monodromy}}$ for sufficiently large $\min_i \nu_i$. In particular, any simple nonabelian G gives rise to full Hurwitz maps of arbitrarily large degrees.

Field of definition: If all C_i are rational, then $\pi_h : \mathrm{Hur}_h \to \mathrm{Conf}_\nu$ is $\boxed{\text{defined over } \mathbb{Q}}$. (For $G = S_n$, all C_i are rational. For general G, if C_i consists of involutions, it is rational.)

Bad reduction. π_h has all its bad reduction $\boxed{\text{within the set } \mathcal{P}_G}$ of primes dividing G.

3D. Belyi pencils

A Belyi pencil of type ν is an embedding $u : \mathrm{P}^1 - \{0, 1, \infty\} \to \mathrm{Conf}_\nu$.
Examples (with $k = j - 1$):

$$\begin{aligned} u_4 : \mathrm{P}^1_j - \{0,1,\infty\} &\to \mathrm{Conf}_{3,1}, \\ j &\mapsto ((t^3 - 3jt + 2j), \{\infty\}), \end{aligned}$$

$$\begin{aligned} u_5 : \mathrm{P}^1_j - \{0,1,\infty\} &\to \mathrm{Conf}_{4,1}, \\ j &\mapsto ((k^2t^4 - 6jkt^2 - 8jkt - 3j^2), \{\infty\}). \end{aligned}$$

Polynomial discriminants are

$$\begin{aligned} D_4(j) &= 2^2 3^3 j^2 (j-1), \\ D_5(j) &= -2^{12} 3^3 j^4 (j-1)^6. \end{aligned}$$

So the bad reduction sets are $\mathcal{P}_{u_4} = \mathcal{P}_{u_5} = \{2, 3\}$.
It is easy to get Belyi pencils into infinitely many Conf_ν defined over $\mathbb{Q}$, all with bad reduction set in any given non-empty $\mathcal{P}$.

3E. Def. and properties of Hurwitz-Belyi maps

Definition

Suppose given

- A Hurwitz parameter $h = (G, C, \nu)$ and
- A Belyi pencil $u : \mathsf{P}^1 - \{0, 1, \infty\} \to \mathrm{Conf}_\nu$.

The corresponding Hurwitz-Belyi map $\beta_{h,u}$ is obtained by pulling back and canonically completing:

$$\begin{array}{ccccc} \mathsf{X} & \supset & \mathsf{X}^0 & \to & \mathrm{Hur}_h \\ \beta_{h,u} \downarrow & & \downarrow & & \downarrow \pi_h \\ \mathsf{P}^1 & \supset & \mathsf{P}^1 - \{0, 1, \infty\} & \stackrel{u}{\to} & \mathrm{Conf}_\nu . \end{array}$$

If h and u are defined over $\mathbb{Q}$, then $\beta_{h,u}$ is likewise $\boxed{\text{rational}}$.

The bad reduction set of $\mathcal{P}_{h,u}$ is contained in $\boxed{\mathcal{P}_G \cup \mathcal{P}_u}$.

If π_h is $\boxed{\text{full}}$ then one would generally expect $\beta_{h,u}$ to be full too. However it's possible that the monodromy group becomes smaller.

3F. Four full examples

All examples $\beta_{h,u} : \mathrm{P}^1_x \to \mathrm{P}^1_j$ are presented via $f(j,x) = 0$.

Example 1. $h = (S_5, (41, 2111), (3,1))$ and $u = u_4$ yields degree $n = 32$ and $\mathcal{P} = \{2,3,5\}$:

$$\begin{aligned} f(j,x) \;=\; & \big(x^{10} - 38x^9 + 591x^8 - 4920x^7 + \\ & \quad 24050x^6 - 71236x^5 + 125638x^4 \\ & \quad -124536x^3 + 40365x^2 + 85050x \\ & \quad -91125\big)^3 \left(x^2 - 14x - 5\right) \\ & +2^{20}3^3 j x^6 (x-5)^5 \left(x^2 - 4x + 5\right)^4 (x-9)^3, \end{aligned}$$

$$\mathrm{disc}_x(f(j,x)) \;=\; -2^{1032}3^{261}5^{289} j^{20} (j-1)^{16}.$$

$\beta_{h,u}$ is one of about 16 full Belyi maps with ramification triple

$$(3^{10}\,1^2,\; 2^{16}, 10\;6\;5\;4^2\;3).$$

Example 2. $h = (A_6, (3111, 2211), (4, 1))$ and $u = u_5$ yields degree $n = 192$ and $\mathcal{P} = \{2, 3, 5\}$:

$$
\begin{aligned}
f(j, x) = \\
&\left(14659268544x^{64} \cdots - 569088x^2 + 11008x - 64\right)^3 \\
&-2^4 3^6 j \left(-3x^3 + 7x^2 - 11x + 1\right)^{15} \\
&\left(-6x^5 + 36x^4 - 72x^3 + 64x^2 - 23x + 4\right)^{12} \cdot \\
&\left(-3x^3 + 9x^2 - 3x - 1\right)^9 (3 - x)^5 x^4 (1 - x)^3 \cdot \\
&\left(9x^8 - 72x^7 + 240x^6 - 444x^5 + 474x^4 - 280x^3 + 72x^2 - 12x + 1\right)^6,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{disc}_x(f(j, x)) = \\
&-2^{6028} 3^{9585} 5^{10525} j^{128} (j - 1)^{84}.
\end{aligned}
$$

$\beta_{h,u}$ is one of about

$$1,900,000,000,000,000,000,000,000,000,000,000$$

full Belyi maps with ramification triple

$$(3^{64},\ 2^{84}\, 1^{24},\ 15^3\, 12^5\, 9^3\, 6^8\, 5\ 4\ 3).$$

Example 3. $h = (G_2(2), (2B, 4D), (3,1))$ and $u = u_4$ yields degree $n = 40$ and $\mathcal{P} = \{2,3,7\}$:

$$\begin{aligned} f(j,x) &= \big(64x^{12} - 576x^{11} + 2400x^{10} - 5696x^9 \\ &\qquad +7344x^8 - 3168x^7 - 4080x^6 \\ &\qquad +8640x^5 - 7380x^4 - 1508x^3 \\ &\qquad +8982x^2 - 7644x + 2401\big)^3 \cdot \\ &\qquad \big(4x^4 - 20x^3 + 78x^2 - 92x + 49\big) \\ &\quad -2^8 3^{12} j \big(2x^2 - 4x + 3\big)^8 x^7 (x-2)^3 (x+1)^2, \end{aligned}$$

$$\operatorname{disc}_x(f(j,x)) = -2^{1148}\, 3^{906}\, 7^{91}\, j^{24}\, (j-1)^{20}.$$

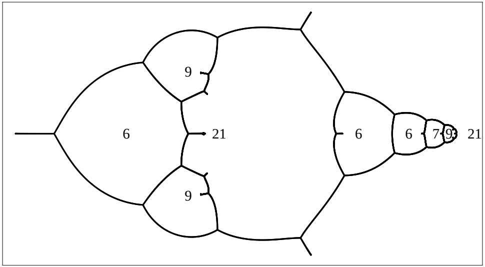
$\beta_{h,u}$ is one of about 10,000 full Belyi maps with ramification triple

$$(3^{12}\,4,\ 2^{20},\ 12\ 8^2\ 7\ 3\ 2).$$

Example 4. $h = (GL_3(2), (22111, 331), (4, 1))$ and $u = u_5$ yields degree $n = 96$ and $\mathcal{P} = \{2, 3, 7\}$:

$$\begin{aligned} f_{96}(j,x) \;&=\; \left(7411887x^{32} \cdots - 3869835264\right)^3 \\ &\quad -2^{20} j \left(7x^2 - 14x + 6\right)^{21} \left(2x^3 - 15x^2 + 18x - 6\right)^9 \cdot \\ &\qquad x^6 \left(x^2 + 2x - 2\right)^6 (3x-2)^2. \end{aligned}$$

The dessin $\beta^{-1}([0,1])$ in P^1_x:



$\beta_{h,u}$ is one of $\approx 3,100,000,000,000,000$ full Belyi maps with triple

$$(3^{32},\ 2^{40}\, 1^{16}, 21^2\, 9^3\, 7\, 6^3\, 2).$$

3G. Towards proving the conjecture

Unlike the parallel assertion about number fields, the conjecture predicting unbounded degrees may be provable by braid monodromy arguments. Polynomial equations, like those of Examples 1-4, are not at all needed.

Verification in degrees well beyond where polynomials are computable is feasible. E.g., for $h = (S_5, (41, 221), (4, 1))$ the degree of π_h is 1440. A braid monodromy calculation shows that β_{h,u_5} still has full monodromy. It would be interesting to take calculations of this sort into much larger degrees.

What is needed to prove the conjecture is theoretical control over the potential drop in monodromy when one passes from the high-dimensional Hurwitz cover π_h to the one-dimensional Belyi cover $\beta_{h,u}$.

Some References

Works in blue are on my homepage `www.davidproberts.net`.

The initial degree 64 example is $U_{8,9}$ from *Chebyshev covers and exceptional number fields*.

The examples with $\mathcal{P} = \{2, 3\}$ are from *Number fields with discriminant $\pm 2^a 3^b$ and Galois group A_n or S_n*, with Gunter Malle, London Mathematical Society Journal of Computation and Mathematics, 8, (2005). 80-101.

The connection with Bhargava's mass heuristic is explained in *Wild partitions and number theory*, Journal of Integer Sequences, Vol. 10, (2007), Article 07.6.6, 34 pages.

The full-monodromy theorem with Venkatesh is Theorem 5.1 in *Hurwitz monodromy and full number fields*, with Akshay Venkatesh, Algebra and Number Theory 9 (2015) no. 3, 511–545.

A standard reference on Hurwitz schemes is *Champs de Hurwitz*, by José Bertin and Matthieu Romagny, Mém. Soc. Math. Fr. 125–126 (2011), 219pp.

The single paper most closely corresponding to this talk is *Hurwitz-Belyi maps*, Pub. Math. de Besançon, Algèbre et Théorie des Nombres, 2018, 25–67.

Example 4 takes as its starting point Theorem 4.2 of *Multi-parameter polynomials with given Galois group*, by Gunter Malle, J. Symbolic Comput. 30 (2000) 717–731.

Monodromy computations in high degree for Hurwitz covers could use *A GAP package for braid orbit computation and applications,* by Kay Magaard, Sergey Shpectorov, and Helmut Völklein, Experiment. Math 12 (2003), no. 4, 385–393.

Monodromy computations after specialization to Belyi pencils could involve ideas from *Galois invariants of dessins d'enfants*, by Jordan Ellenberg, 27–42, Proc. Sympos. Pure Math., 70 (2002).