Quadratic relations between Feynman integrals

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(reporting on joint work
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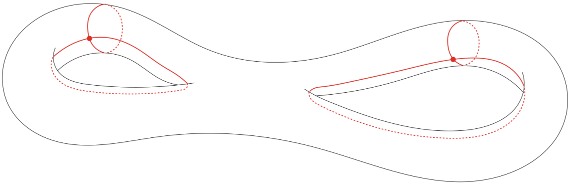
January 16, 2019

Background: Periods in the context of curves

On a genus g curve X over $\mathbb{Q}$, one has $2g$ independent one-cycles Δ_u and $2g$ independent one-forms ω_a and three $2g$-by-$2g$ matrices:

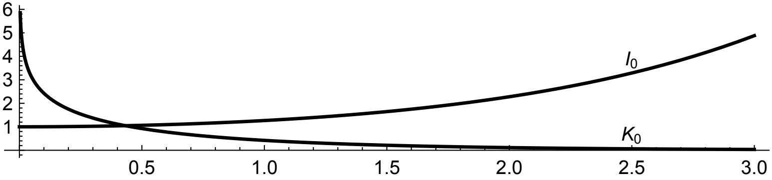
$$F = \left(\int_{\gamma_u} \omega_a \right), \qquad B = (\Delta_u \cdot \Delta_v), \qquad D = \left(\int_X \omega_a \wedge \overline{\omega}_b \right).$$

The periods satisfy the quadratic relations $\boxed{FDF^t = B}$ with D and B having rational entries. For genus two, $B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.



Some interesting integrals

Classical Bessel functions I_0 and K_0 on the interval $(0, \infty)$ are given by various equivalent formulas and have simple graphs:



The integrals

$$\mathrm{Int}(a, b, c) = \int_0^\infty I_0(x)^a K_0(x)^b x^c \, dx.$$

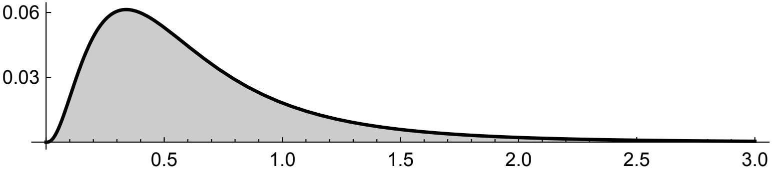
are important because they arise in quantum field theory from Feynman diagrams.

A representative integral evaluated numerically

For example,

$$\mathrm{Int}(4,5,6) = \int_0^\infty I_0(x)^4 K_0(x)^5 x^6 dx$$

gives the area of



and evaluates to

$$0.0469368415316903315940262851804465389341418080955\ldots$$

Very brief timeline

Pre-2017. Broadhurst [Bro] and others studied the $\mathrm{Int}(a, b, c)$ because of their role in quantum field theory. A focus was on numerically identifying some of the $\mathrm{Int}(a, b, c)$ with special values of L-functions coming from modular forms. There were various strong hints of a general theory, e.g. a paper of Zhi-wei Yun. [Yun]

2017. Broadhurst and I conjectured two general formulas involving all $\mathrm{Int}(a, b, c)$, and supported these conjectures by high precision numeric computations. [BR1, just four pages!] concerns a general L-function formula and [BR2, just one relevant page!] concerns quadratic relations among the $\mathrm{Int}(a, b, c)$. This talk is about [BR2].

2018. Javier Fresán, Claude Sabbah, and Jeng-Daw Yu theoretically established our L-function conjecture [FSY] and are establishing our quadratic relations conjecture in a sequel.

Feynman matrices

The integrals

$$\mathrm{Int}(a,b,c) = \int_0^\infty I_0(x)^a K_0(x)^b x^c \, dx.$$

are naturally grouped according to $N := a + b$, and in this talk we'll take N odd. Put $k = (N-1)/2$ and define a k-by-k matrix F_N with entries

$$F_N(u,a) = \frac{(-1)^{a-1}}{\pi^u} \mathrm{Int}(k+1-u, k+u, 2a-1).$$

For example,

$$F_7 \approx \begin{pmatrix} 0.3566864959 & -0.04118556521 & 0.06789824820 \\ 0.2568847021 & -0.00858666702 & 0.00295037871 \\ 0.2361144815 & -0.00316606434 & 0.00048152310 \end{pmatrix}.$$

Feynman matrices as period matrices

For various reasons, we expected that F_N is a period matrix on a algebraic variety X over $\mathbb{Q}$ of dimension $d = N - 3$. This means that the entries would have an alternate expression

$$F_N(u, a) = \iint \dots \iint_{\Delta_u} \omega_a.$$

Here $\Delta_1, \dots, \Delta_k$ are topological d-cycles in the $2d$-dimensional real manifold X, and $\omega_1, \dots, \omega_k$ are algebraically defined d-forms on this manifold. Define k-by-k matrices B_N and D_N by

$$B_N(u, v) = \Delta_u \cdot \Delta_v, \qquad D_N(a, b) = \int_X \omega_a \wedge \overline{\omega}_b.$$

From the definitions, one would have $\boxed{F_N D_N F_N^t = B_N.}$ The Betti matrix B_N and the de Rham matrix D_N would have rational entries. They would be symmetric in our current case of N odd, and antisymmetric in the other case of N even.

Numerically solving $F_N D_N F_N^t = B_N$

In general, Betti matrices and de Rham matrices have special structures which we expected made some entries 0 in our hoped-for matrices B_N and D_N.

For N small, we found always a unique appropriately normalized numeric solution, e.g.

$$B_7 = \frac{1}{2^6 7}\begin{pmatrix} 84 & 0 & 42 \\ 0 & -35 & 0 \\ 42 & 0 & 30 \end{pmatrix}, \quad D_7 = \frac{1}{2^8}\begin{pmatrix} 816 & 20712 & 11025 \\ 20712 & 11025 & 0 \\ 11025 & 0 & 0 \end{pmatrix}.$$

The block structure on the B_N comes from the fact that alternate Δ_u are either fixed or negated by complex conjugation. The triangular structure on the D_N comes from the fact that the ω_a sit in the smallest possible flags in the Hodge filtration, and all subquotients of this filtration have size zero or one.

Extrapolating to general formulas for B_N and D_N

1. It was easy to see a pattern in the B_N. For odd $N = 2k + 1$, our conjectural formula is

$$B_N(u, v) \;=\; (-1)^{u+k} 2^{-2k-2} (k+u)!(k+v)! \frac{|\text{Bernoulli}_{u+v}|}{(u+v)!}.$$

2. The pattern was much harder to see in the D_N, but eventually we found the inductive formula presented in [BR2].

3. With all three matrices now with independent definitions, we confirmed for much larger N that $F_N D_N F_N^t = B_N$ holds to high precision.

References

[Bro] David Broadhurst. *Feynman integrals, L-series and Kloosterman moments.* Communications in Number Theory and Physics 10 (2016) 527-569.

[BR1] David Broadhurst and David P. Roberts. *L-Series and Feynman integrals.* Matrix Annals 2018. 4 pages.

[BR2] David Broadhurst and David P. Roberts. *Quadratic relations between Feynman integrals.* Loops and Legs in Quantum Field Theory 2018. Proceedings of Science. 8 pages.

[FSY] Javier Fresán, Claude Sabbah, Jeng-Da Yu. *Hodge theory of Kloosterman Connections.* `arXiv:1810.06454`, 69 pages.

[Yun] Zhiwei Yun. *Galois representations attached to moments of Kloosterman sums and conjectures of Evans.* Composita Mathematica 151 (2015), no. 1, 68-120.